Course 2328 Complex Analysis

Due: Friday, at the end of the lecture

Exercise 1

Is there an example of a real-differential function f on $\mathbb{C}$ such that:

(i) f is complex-differentiable at z if and only if $z = 0$ or $z = 1$?

(ii) f is complex-differentiable at z if and only if $|z| \leq 1$?

Solutions

(i) $f(z) = |z(z-1)|^2$ is an example, because both partial derivatives are zero at 0 and 1. On the other hand, since $f(z)$ is real-valued, if the Cauchy-Riemann equations hold at any other point, then all partial derivatives would be zero there.

(ii) Consder f vanishing for $|z| \leq 1$ and satisfying $f(z) = (|z|-1)^2$ for $|z| \geq 1$. Then all partial derivatives vanish on the unit circle, implying real differentiability. The same argument as above implies that f is nowhere complex-differentiable outside the unit disk.

Exercise 2

Use the Cauchy-Riemann equations to decide which of the following functions are holomorphic:

$$(\mathsf{Re} z)^2, \quad |z|^2, \quad -i\bar{z}^2, \quad e^{-z}, \quad e^{-\bar{z}}.$$

Exercise 3

Let $f\colon \Omega \to \mathbb{C}$ be holomorphic. Define the new function $\bar{f}$ by $\bar{f}(z) := \overline{f(\bar{z})}$. Show that $\bar{f}$ is holomorhic on the open set $\bar{\Omega} := \{\bar{z} : z \in \Omega\}$.

Exercise 4

Using the Cauchy-Riemann equations, show:

(i) if a holomorphic function f satisfies $\mathsf{Re} f = \text{const}$, then $f = \text{const}$.

(ii) if $f = u + iv$ is holomorphic and $a, b \in \mathbb{C} \setminus \{0\}$ are such that $au + bv = \text{const}$, then again $f = \text{const}$.

Exercise 5

(i) Show that $(e^z)' = e^z$;

(ii) Show that $\mathsf{cos} z$ and $\mathsf{sin} z$ are $\mathbb{C}$-differentiable and compute their derivatives.