February 16, 2018 Lecturer Dmitri Zaitsev Hilary Term 2018

Course 2328 Complex Analysis

Due: Friday, at the end of the lecture

Exercise 1

Find all z such that the sequence (a_n) is convergent:

(i) $a_n := z^n$;
(ii) $a_n := z^n/n$;
(iii) $a_n := z^n/(n!)$;
(iv) $a_n := \sin(z^n)$;
(v) $a_n := \mathrm{Log}(1-z^n)$ (where Log is the principal value of $\log$);

Solutions

(i) $|z| < 1$ or $z = 1$;

(ii) $|z| \leq 1$;

(iii) For any z.

(iv) For $|z| < 1$, z^n converges, hence a_n does. For $|z| > 1$, $\mathrm{Im} z \neq 0$, a_n diverges, since there exist powers z^n with $\mathrm{Im} z^n$ arbitrarily large, and since $\sin$ grows exponentially in the imaginary direction. (Additional arguments are required to cover the remaining cases.)

(v) As above, convergence for $|z| < 1$. For $|z| > 1$, we have $z^n \to \infty$ and hence $|\mathrm{Log}(1-z^n)| = \ln|1-z^n| \to \infty$. (Additional arguments are required to cover the remaining cases.)

Exercise 2

For what z does the series converge:

(i) $\sum_n z^{2^n - n}$,
(ii) $\sum_n \frac{z^{n^2}}{n^4}$,
(iii) $\sum_n \frac{1}{z^n - n^2}$.

Solutions

(i) For $|z| \geq 1$ the series diverges because it fails the n-th term test. For $|z| < 1$ it converges as follows from the comparison test.

(ii) Here comparison test with $\sum \frac{1}{n^4}$ yields the convergence for $|z| \leq 1$, whereas divergence holds for other z by the same argument as above.

(iii) For $|z| \leq 1$, z^n is dominated by n^2, so the convergence follows from the comparison test with $\sum \frac{1}{n^2}$. On the other hand, for $|z| > 1$, the power z^n dominates and the convergence follows again from the comparison with $\sum \frac{1}{z^n}$.

Exercise 3

Consider the sequence of holomorphic functions $f_n(z) = z - \frac{1}{n}$.

(i) Is the sequence (f_n) converging uniformly on $\mathbb{C}$?

(ii) Is the sequence of squares (f_n^2) converging uniformly on $\mathbb{C}$?

Justify your answer.

Solutions

(i) Yes, because $|f_n(z) - z| \to 0$ uniformly.

(ii) No, because

$$\sup\{z \in \mathbb{C} : |f_n(z)^2 - z^2|\} = \infty.$$

Exercise 4

Determine all points $z \in \mathbb{C}$ where the function $f(z)$ has a limit:

(i) $f(z) = \frac{1}{z} + \frac{1}{z^2}$;

(ii) $f(z) = \frac{(\mathsf{Re} z)^2}{z}$;

(iii) $f(z) = \frac{\mathsf{Im} z}{z^3}$;

(iv) $f(z) = \mathsf{Log} z$.

Justify your answer.

Solutions

(i) Any $z \neq 0$.

(ii) Any $z \neq 0$. There is no limit at 0 because the function has different limits along different rays.

(iii) Any $z \neq 0$.

(iv) $\mathsf{Log} z$ is defined away from the negative ray $\mathbb{R}_{\leq 0}$, at which points there is no limit because for $z \neq 0$, the limits from above and below are different, and for $z = 0$ there is no limit.

Exercise 5

For which $w \in \mathbb{C}$, the power function $f(z) := z^w$ has a continuous branch in the closed upper-half plane $\{\mathsf{Im} z \geq 0\}$?

Solutions

The power is defined as

$$z^w := e^{w \mathsf{log} z},$$

whose branches are determined by the branches of $\mathsf{log} z$. Any branch of $\mathsf{log} z$ in the open upper-half plane $\mathsf{Im} z > 0$ extends continuously to the closure minus 0. Thus it remains to verify the continuity at 0. For instance, any integer power is continuous at 0.

In general

$$|z^w| = e^{\mathsf{Re}(w \mathsf{log} z)},$$

which for $w \in \mathbb{R}$ equals

$$|z^w| = e^{w \ln |z|} = |z|^w \to 0, \quad z \to 0.$$

On the other hand, for $w = a + ib \notin \mathbb{R}$, the imaginary part enters with the argument:

$$|z^w| = e^{\mathsf{Re}((a+ib) \mathsf{log} z)} = e^{a \ln |z| - b \arg z}$$

and the $\arg z$ is constant along rays but has value depending on the ray. Hence in that case the power has no limit at 0, and the final answer is that no branch is possible.

Exercise 6

Prove or disprove:

(i) A finite union of compact subsets in $\mathbb{C}$ is compact.
(ii) Arbtitrary union of compact subsets in $\mathbb{C}$ is compact.
(iii) A finite union of connected subsets in $\mathbb{C}$ is connected.
(iv) Arbtitrary union of connected subsets in $\mathbb{C}$ is connected.
(v) Arbtitrary union of connected subsets in $\mathbb{C}$ having a nonempty intersection is connected.

Solutions

Only (i) and (v) are true.