Course 2328 Complex Analysis

Sheet 5

Due: Friday, at the end of the lecture

Exercise 1

Let γ be the sum of two line segments connecting -1 with iy and iy with 1, where y is a fixed parameter.

(i) Write an explicit parametrization for γ;

(ii) For every y, evaluate the integrals $\int_\gamma z\,dz$ and $\int_\gamma \bar{z}\,dz$. Which of the integrals is independent of y?

(iii) Use (ii) to show that the conclusion of Cauchy's theorem does not hold for $f(z) = \bar{z}$.

Exercise 2

(i) Calculate $\int_\gamma f(z)\,dz$, where

$$f(z) = z + \frac{1}{z} - \frac{4}{z^3}$$

and $\gamma(t) = ce^{it}$, $0 \le t \le 2\pi$.

(ii) Use (i) to show that $f(z)$ does not have an antiderivative in its domain of definition.

(iii) Does $f(z) = \frac{1}{z^n}$ have an antiderivative, where $n \ge 2$ is an integer?

(iv) Give an example of an open set Ω, where the function $f(z) = \frac{1}{(z+1)(z-1)}$ does not have an antiderivative.

Justify your answer.

Exercise 3

Calculate the residues:

(i) $\mathrm{Res}_3 \frac{z^2-e^{-z}}{z-3}$;

(ii) $\mathrm{Res}_0 \frac{\sin(z^2)-e^{2z}}{z^5+z^3-z}$;

(iii) $\mathrm{Res}_{-1} \frac{\cos(2\pi z^{-2})+ze^z}{\sin(\pi z)}$.

Exercise 4

Evaluate the integrals:

(i) $\int_{|z|=2} \frac{e^z}{(z^2+z)(z-3)}\, dz$;

(ii) $\int_0^{2\pi} \frac{1}{(2-\sin\theta)(3+\cos\theta)} d\theta$;

(iii) $\int_{-\infty}^{\infty} \frac{x^3-x}{x^6+1} dx$;

(iv) $\int_{-\infty}^{\infty} \frac{x^3 e^{i\lambda x}}{x^4+1} dx$, $\lambda > 0$;

(v) $\int_0^{\infty} \frac{x^\alpha}{x^2+1} dx$, $-1 < \alpha < 1$, $\alpha \notin \mathbb{Z}$.