February 26, 2017 Lecturer Dmitri Zaitsev Hilary Term 2017

Course 2328 Complex Analysis

Sheet 2

Due: Friday, at the end of the lecture on Thursday of the next week

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Exercise 1

Using the definition show:

(i) Finite intersections and arbitrary unions of open sets are open.

(ii) Finite unions and arbitrary intersections of closed sets are closed.

(iii) Arbitrary union of connected sets containing a common point is connected.

Exercise 2

(i) If Ω is open in $\mathbb{C}$ and $A \subset \mathbb{C}$ any subset, then $\Omega \cap A$ is open in A.

(ii) If $A \subset \mathbb{C}$ is any subset and U is open in A, then there exists Ω open in $\mathbb{C}$ with

$$\Omega \cap A = U.$$

Hint. Use the definition and construct Ω as union of disks.

Exercise 3

Give examples of:

(i) Infinite intersections of open sets that is not open.

(ii) Infinite union of closed sets that is not closed.

(iii) Intersection of two connected sets containing a common point that is not connected.

Exercise 4

For what z does the series converge:

(i) $\sum_n z^{n+3^n}$,

(ii) $\sum_n \frac{z^{n^2}}{n^3}$,

(iii) $\sum_n \frac{1}{z^n - n}$.

Which are power series? Justify your answer.

Exercise 5

Consider the sequence of holomorphic functions $f_n(z) = z + \frac{1}{n^2}$.

(i) Is the sequence (f_n) converging uniformly on $\mathbb{C}$?

(ii) Is the sequence of squares (f_n^2) converging uniformly on $\mathbb{C}$?

Justify your answer.

Exercise 6

Use the Cauchy-Riemann equations to decide which of the following functions are holomorphic:

$$(\mathsf{Im} z)^2, \quad |z|^2, \quad i\bar{z}^2, \quad e^{2z}, \quad e^{\bar{z}}.$$

Exercise 7

Let $f\colon \Omega \to \mathbb{C}$ be holomorphic. Define the new function $\bar{f}$ by $\bar{f}(z) := \overline{f(\bar{z})}$. Show that $\bar{f}$ is holomorhic on the open set $\bar{\Omega} := \{\bar{z} : z \in \Omega\}$.

Exercise 8

Using the Cauchy-Riemann equations, show:

(i) if a holomorphic function f satisfies $\mathsf{Im} f = \text{const}$, then $f = \text{const}$.

(ii) if $f = u + iv$ is holomorphic and $a, b \in \mathbb{C} \setminus \{0\}$ are such that $au + bv = \text{const}$, then again $f = \text{const}$.

Exercise 9

(i) Show that $(e^z)' = e^z$. (Hint. Differentiate in the direction of the x-axis.)

(ii) Let f be any branch of $\mathsf{log} z$ (defined in an open set). Using the fact that f is inverse to e^z, show that f is holomorphic and $f'(z) = \frac{1}{z}$.

Exercise 10

Is there an example of a real-differential function f on $\mathbb{C}$ such that:

(i) f is complex-differentiable at z if and only if $z = 0$?

(i) f is complex-differentiable at z if and only if $|z| \leq 1$?