COMPLEX ANALYSIS: LECTURE NOTES

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Contents

1. The origin of complex numbers

1.1. **Solving quadratic equation.** The simplest quadratic equation having no real solutions is

$$x^2 + 1 = 0.$$

Trying to solve it, we have to introduce the symbol $i = \sqrt{-1}$. As we also want to add and multiply, we are lead to consider expressions

$$z = a + ib,$$

where a and b are arbitrary *real numbers*. We call z a *complex number* and write

$$a = \mathsf{Re}\, z, \quad b = \mathsf{Im}\, z$$

for the *real* and *imaginary* parts of z.

Introducing i may appear artificial and is not relevant to the original problem of finding *real* solutions. In fact, solution of quadratic equation was known to the ancient Greeks, who never came across complex numbers nor felt any need for it.

1.2. **Cubic equation and Cardano's formula.** In contrast to quadratic equation, solving a cubic equation even over reals forces you to pass through complex numbers. In fact, this is how complex numbers were discovered.

Consider the general cubic equation

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \quad a_3 \neq 0.$$

Dividing by a_3 we can reduce it to

$$x^3 + a_2x^2 + a_1x + a_0 = 0.$$

Using binomial formula $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, we can rewrite it as

$$(x + a_2/3)^3 + p(x + a_2/3) + q = 0$$

for suitable p and q, or after a change of variables,

$$x^3 + px + q = 0. \tag{1.1}$$

Cardano's formula for solving (1.1) is

$$x = u + v, \quad u = \sqrt[3]{-\frac{q}{2} + \sqrt{D}}, \quad v = \sqrt[3]{-\frac{q}{2} - \sqrt{D}}, \quad D = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3,$$

such that u, v satisfy

$$uv = -\frac{p}{3}. \tag{1.2}$$

Condition (1.2) is sometimes forgotten when Cardano's formula is stated, but is essential and cannot be dropped, or else the formula gives other values of x that are not among solutions.

The main difference from the quadratic equation formula is the fact that, in Cardano's formula, we may need to pass in our calculations through complex numbers even if the final result is real. We illustrate this by a simple example.

1.3. **Example of using Cardano's formula.** Consider the cubic equation

$$x^3 - 3x = 0.$$

Since $x^3 - 3x = x(x - \sqrt{3})(x + \sqrt{3})$, we have 3 real solutions.

Now in Cardano's formula we have $p = -3$, $q = 0$, $D = -1$, and hence

$$u = \sqrt[3]{\sqrt{-1}}, \quad v = \sqrt[3]{-\sqrt{-1}}.$$

Thus, even if all solutions are real, we need to evaluate $\sqrt{-1}$! We shall now introduce the necessary tools to complete the calculation.

2. Algebraic operations with complex numbers

2.1. **Addition and multiplication.** Using the rule $i^2 = -1$ we can define *addition* and *multiplication* of complex numbers:

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2),$$

$$(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2) + i(a_1b_2 + a_2b_1) + i^2(b_1b_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1).$$

2.2. **The complex conjugate.** The *complex conjugate* of $z = a + ib$ is given by

$$\bar{z} = a - ib,$$

from where we have formulas for the real and imaginary parts:

$$a = \operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad b = \operatorname{Im} z = \frac{z - \bar{z}}{2i}.$$

We also have the important formula

$$z\bar{z} = (a + ib)(a - ib) = a^2 + b^2. \tag{2.1}$$

The conjugation defines an *automorphism* in the sense that

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2.$$

2.3. **Division.** We can use conjugates and formula (2.1) for division:

$$\frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} = \frac{(a_1a_2 + b_1b_2) + i(a_2b_1 - a_1b_2)}{a_2^2 + b_2^2}.$$

Thus we can always divide by a complex number different from zero.

It is now easy to see that complex numbers form a field, denoted by $\mathbb{C}$.

3. The polar form

Any complex number $z = a + ib$ can be written in its *polar form* as

$$z = r(\cos\theta + i\sin\theta),$$

where

$$r = |z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}} \geq 0$$

is the *modulus* or *absolute value* of z and θ is the *argument* of z, defined for $z \neq 0$ and satsifying

$$\cos\theta = \frac{a}{r}, \quad \sin\theta = \frac{b}{r}, \quad \tan\theta = \frac{b}{a} \text{ (for } a \neq 0).$$

For $z = 0$, the argument θ is not defined. For $z \neq 0$, it is defined up to a multiple of 2π. In particular, for $z \neq 0$, we can always choose θ uniquely in the interval $(-\pi, \pi]$, which we call the *principal argument* and denote by $\operatorname{Arg} z$. By $\arg z$ we denote the set of all possible arguments, i.e.

$$\arg z = \{\operatorname{Arg} z + 2\pi k : k \in \mathbb{Z}\}.$$

Using the *Euler's formula*

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3.1}$$

we can rewrite the polar form as

$$z = re^{i\theta}.$$

In polar form, multiplication and division look particularly simple:

$$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}, \quad \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}.$$

The first formula can be iterated to obtain *de Moivre's formula*:

$$(re^{i\theta})^n = r^ne^{in\theta},$$

which can be used for calculating the nth roots:

$$\sqrt[n]{re^{i\theta}} = \sqrt[n]{r}e^{i\frac{\theta+2\pi k}{n}} = \sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right), \quad k = 0, \dots, n-1.$$

Note that θ is always defined up to a multiple of 2π and hence the addition of $2\pi k$ is essential in the formula, but it suffices to take only finitely many values $k = 0, \dots, n-1$, to obtain all n roots.

3.1. **Completing our example of using Cardano's formula.** We can now complete our calculation in the example of using Cardano's formula in §1.3. We have the polar form

$$i = e^{i\frac{\pi}{2}}$$

and hence

$$u = \sqrt[3]{i} = \cos\frac{\pi/2 + 2\pi k}{3} + i\sin\frac{\pi/2 + 2\pi k}{3}, \quad k = 0, 1, 2.$$

Calculating we obtain the values

$$u = \frac{\sqrt{3}}{2} + i\frac{1}{2}, \quad -\frac{\sqrt{3}}{2} + i\frac{1}{2}, \quad -i.$$

Similarly,

$$v = \sqrt[3]{-i} = \cos\frac{-\pi/2 + 2\pi k}{3} + i\sin\frac{-\pi/2 + 2\pi k}{3}, \quad k = 0, 1, 2,$$

which yields

$$v = \frac{\sqrt{3}}{2} - i\frac{1}{2}, \quad -\frac{\sqrt{3}}{2} - i\frac{1}{2}, \quad i.$$

Finally, according to the rule (1.2), we only consider $x = u + v$ for u, v satisfying $uv = -p/3$, i.e. $uv = 1$ in our case. Hence we can only combine the first, second and third values of u with the first, second and third values of v respectively, which yields the three solutions

$$x = u + v = \sqrt{3}, \quad -\sqrt{3}, \quad 0.$$

Note that without the rule (1.2), we would e.g. add the first value of u with the third value of v giving

$$x = \frac{\sqrt{3}}{2} + i\frac{3}{2},$$

which is not a solution of our equation (1.1).

4. Elementary topology

4.1. **Open sets.** In the sequel denote by

$$\Delta_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$$

the disk with center $a \in \mathbb{C}$ and radius $r > 0$. A subset $U \subset \mathbb{C}$ is *open (in $\mathbb{C}$)* if every point $z_0 \in U$ can be surrounded by disk $\Delta_\varepsilon(z_0) \subset U$ for some $\varepsilon > 0$. More generally, a subset $A \subset S$ is *open in the set $S \subset \mathbb{C}$* (briefly *open in S*, also *open in the relative topology of S*) if for every $z_0 \in A$, there exists $\varepsilon > 0$ with

$$\Delta_\varepsilon(z_0) \cap S \subset A.$$

It is straightforward to check that arbitrary unions $\bigcup_{\alpha \in A} U_\alpha$ and finite intersections $U_1 \cap \cdots \cap U_n$ of open sets (in a set S) are open (in a set S).

4.2. **Closed sets.** A set $F \subset S$ is *closed in S* if $S \setminus F$ is open in S. Using analogous properties of open sets it is easy to see that arbitrary intersections and finite unions of closed sets in S are closed in S.

4.3. **Connected sets.** A set $S \subset \mathbb{C}$ is *connected* if it cannot be covered by disjoint open sets U and V such that $U \cap S \neq \emptyset$, $V \cap S \neq \emptyset$. Equivalently, one can use open sets in S:

Lemma 4.1. *S is connected if and only if it cannot be covered by nonempty disjoint sets A and B, which are open in S.*

5. Limits of sequences and functions

5.1. **Limit of sequence of complex numbers.** A sequence of complex numbers (z_n) converges to z_0, written

$$z_0 = \lim_{n \to \infty} z_n, \text{ or } z_n \to z_0, \ n \to \infty,$$

if the distance $|z_n - z_0|$ converges to 0. The latter is equivalent to the real and imaginary parts $\mathsf{Re}\, z_n$, $\mathsf{Im}\, z_n$ converging to $\mathsf{Re}\, z_0$, $\mathsf{Im}\, z_0$ respectively.

5.2. **Cauchy's criterion.** A sequence (z_n) is convergent (converges to some limit) if and only if for every $\varepsilon > 0$, there is N such that for all $m, n \geq N$, one has $|z_m - z_n| < \varepsilon$. This criterion for complex sequences can be derived from the analogous criterion for real sequences.

5.3. **Limits of functions.** Limits of functions can be considered for functions defined on subsets $S \subset \mathbb{C}$ at their *limit points*. Recall that a point $a \in \mathbb{C}$ is a limit point of S if every punctured disk $\Delta_\varepsilon(a) \setminus \{a\}$ contains a point of S (which implies that every punctured disk contains infinitely many points of S). Note that a limit point of S does not need to belong to S.

If a is a limit point of S, then L is the limit of a function $f \colon S \to \mathbb{C}$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $z \in \Delta_\delta(a) \cap S$ implies $|f(z) - L| < \varepsilon$.

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