March 25, 2016 Lecturer Dmitri Zaitsev Hilary Term 2016

Course 2328 Complex Analysis I

Due: Thursday, at the end of the lecture

Exercise 1

Is there an example of a real-differential function f on $\mathbb{C}$ such that:

(i) f is complex-differentiable at z if and only if $z = 0$?

(i) f is complex-differentiable at z if and only if $|z| \leq 1$?

Exercise 2

Let γ be the sum of two line segments connecting -2 with iy and iy with 2, where y is a fixed parameter.

(i) Write an explicit parametrization for γ;

(ii) For every y, evaluate the integrals $\int_\gamma z\,dz$ and $\int_\gamma \bar{z}\,dz$. Which of the integrals is independent of y?

(iii) Use (ii) to show that the conclusion of Cauchy's theorem does not hold for $f(z) = \bar{z}$.

Exercise 3

(i) Calculate $\int_\gamma f(z)\,dz$, where $f(z) = \frac{1}{z}$ and $\gamma(t) = 2e^{it}$, $0 \leq t \leq 2\pi$.

(ii) Use (i) to show that $f(z)$ does not have an antiderivative in its domain of definition.

(iii) Does $f(z) = \frac{1}{z^n}$ have an antiderivative, where $n \geq 2$ is an integer?

(iv) Give an example of an open set Ω, where the function $f(z) = \frac{1}{(z+1)(z-1)}$ does not have an antiderivative.

Justify your answer.

Exercise 4

Calculate the residues:

(i) $\mathrm{Res}_0 \frac{\cos(z^3) - 2e^z}{z^8 - z}$;

(ii) $\mathrm{Res}_{-1} \frac{\cos(2\pi z^2) + z}{\sin(\pi z)}$.

Exercise 5

Evaluate the integrals:

(i) $\int_{-\infty}^{\infty} \frac{x^4}{x^8+1} dx$.

(ii) $\int_{-\infty}^{\infty} \frac{x+x^2}{x^4+2x^2+2} dx$.