Exercise 1

Using the definition show:

(i) Finite intersections and arbitrary unions of open sets are open.
(ii) Finite unions and arbitrary intersections of closed sets are closed.
(iii) Arbitrary union of connected sets containing a common point is connected.

Exercise 2

(i) If Ω is open in \( \mathbb{C} \) and \( A \subset \mathbb{C} \) any subset, then \( \Omega \cap A \) is open in \( A \).
(ii) If \( A \subset \mathbb{C} \) is any subset and \( U \) is open in \( A \), then there exists \( \Omega \) open in \( \mathbb{C} \) with

\[ \Omega \cap A = U. \]

**Hint.** Use the definition and construct \( \Omega \) as union of disks.

Exercise 3

Give examples of:

(i) Infinite intersections of open sets that is not open.
(ii) Infinite union of closed sets that is not closed.
(iii) Intersection of two connected sets containing a common point that is not connected.

Exercise 4

Consider the sequence of holomorphic functions \( f_n(z) = z^2 - \frac{1}{n} \).

(i) Is the sequence \((f_n)\) converging uniformly on \( \mathbb{C} \)?
(ii) Is the sequence of squares \((f_n^2)\) converging uniformly on \( \mathbb{C} \)?
(iii) Is the sequence of squares \((f_n^2)\) converging uniformly on the unit disk in \( \mathbb{C} \)? Justify your answer.
Exercise 5
For which $z$ does the sequence converge:
(i) $f_n(z) = e^{nz}$;
(ii) $f_n(z) = \sin(nz)$.

Exercise 6
For which $z$ does the series converge:
(i) $\sum_n z^{n^2}$,
(ii) $\sum_n \frac{z^{n+n^2}}{n}$,
(iii) $\sum_n \frac{1}{z^n-n}$.
Which are power series? Justify your answer.

Exercise 7
Let $f: \Omega \to \mathbb{C}$ be holomorphic. Define the new function $\bar{f}$ by $\bar{f}(z) := \overline{f(z)}$. Show that $\bar{f}$ is holomorphic on the open set $\bar{\Omega} := \{ \bar{z} : z \in \Omega \}$.

Exercise 8
Use the Cauchy-Riemann equations to decide which of the following functions are holomorphic:
$(\text{Im}z)^2, \quad i\bar{z}, \quad \bar{z}^2 + z, \quad e^{10z}, \quad e^\bar{z}$.

Exercise 9
Using the Cauchy-Riemann equations, show:
(i) if a holomorphic function $f$ satisfies $\text{Im} f = \text{const}$, then $f = \text{const}$.
(ii) if $f = u + iv$ is holomorphic and $a, b \in \mathbb{C} \setminus \{0\}$ are such that $au + bv = \text{const}$, then again $f = \text{const}$.