October 31, 2014 Lecturer Dmitri Zaitsev Michaelmas Term 2014

Course 2325 Complex Analysis I

S h e e t 2

Due: at the end of the lecture on Wednesday of the next week

Exercise 1

For what z does the series converge:

(i) $\sum_n z^{n+2^n}$,

(ii) $\sum_n \frac{z^{n^2}}{n^2}$,

(iii) $\sum_n \frac{1}{z^n-1}$.

Which are power series? Justify your answer.

Exercise 2

Consider the sequence of holomorphic functions $f_n(z) = z + \frac{1}{n}$.

(i) Is the sequence (f_n) converging uniformly on $\mathbb{C}$?

(ii) Is the sequence of squares (f_n^2) converging uniformly on $\mathbb{C}$?

Justify your answer.

Exercise 3

Use the Cauchy-Riemann equations to decide which of the following functions are holomorphic:

$$(\mathsf{Re} z)^2, \quad i|z|^2, \quad \bar{z}^2, \quad e^{2z}, \quad e^{\bar{z}}.$$

Exercise 4

Let $f\colon \Omega \to \mathbb{C}$ be holomorphic. Define the new function $\bar{f}$ by $\bar{f}(z) := \overline{f(\bar{z})}$. Show that $\bar{f}$ is holomorhic on the open set $\bar{\Omega} := \{\bar{z} : z \in \Omega\}$.

Exercise 5

Using the Cauchy-Riemann equations, show:

(i) if a holomorphic function f satisfies $\mathsf{Re} f = \text{const}$, then $f = \text{const}$.

(ii) if $f = u + iv$ is holomorphic and $a, b \in \mathbb{C} \setminus \{0\}$ are such that $au + bv = \text{const}$, then again $f = \text{const}$.

Exercise 6

(i) Show that $(e^z)' = e^z$. (Hint. Differentiate in the direction of the x-axis.)

(ii) Let f be any branch of $\mathsf{log} z$ (defined in an open set). Using the fact that f is inverse to e^z, show that f is holomorphic and $f'(z) = \frac{1}{z}$.