Exercise 1

(i) Show that \((e^z)' = e^z\). (Hint. Differentiate in the direction of the \(x\)-axis.)

(ii) Let \(f\) be any branch of \(\log z\) (defined in an open set). Using the fact that \(f\) is inverse to \(e^z\), show that \(f\) is holomorphic and \(f'(z) = \frac{1}{z}\).

Exercise 2

Let \(\gamma\) be the sum of two line segments connecting \(-2\) with \(iy\) and \(iy\) with \(2\), where \(y\) is a fixed parameter.

(i) Write an explicit parametrization for \(\gamma\);

(ii) For every \(y\), evaluate the integrals \(\int_\gamma z\,dz\) and \(\int_\gamma \bar{z}\,dz\). Which of the integrals is independent of \(y\)?

(iii) Use (ii) to show that the conclusion of Cauchy’s theorem does not hold for \(f(z) = \bar{z}\).

Exercise 3

(i) Calculate \(\int_\gamma f(z)\,dz\), where \(f(z) = \frac{1}{z}\) and \(\gamma(t) = e^{it}, 0 \leq t \leq 2\pi\), is the unit circle.

(ii) Use (i) to show that \(f(z)\) does not have an antiderivative in its domain of definition.

(iii) Does \(f(z) = \frac{1}{z^n}\) have an antiderivative, where \(n \geq 2\) is an integer?

(iv) Give an example of a domain \(\Omega\), where the function \(f(z) = \frac{1}{z(z-1)}\) does not have an antiderivative.

Justify your answer.

Exercise 4

Calculate the residues:

(i) \(\text{Res}_0 \frac{\sin(z^3) - e^z}{z^4 + e^z + z}\);

(ii) \(\text{Res}_1 \frac{\cos(2\pi z^2) + z}{e^z - e}\).

Exercise 5

Evaluate the integrals:
(i) \[ \int_{|z| = 4} \frac{e^z}{(z^2 - z)(z + 5)} \, dz; \]
(ii) \[ \int_0^{2\pi} \frac{\sin^2 \theta}{3 - \cos \theta} \, d\theta; \]
(iii) \[ \int_{-\infty}^{\infty} \frac{x - x^2}{x^4 - 2x^2 + 2} \, dx; \]
(iv) \[ \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^4 - 2x^2 + 2} \, dx, \quad \lambda > 0. \]