November 28, 2012 Lecturer Dmitri Zaitsev Michaelmas Term 2012

Course 2325 2012 Complex Analysis I

Sheet 3

Due: at the end of the lecture on Wednesday next week

Exercise 1

(i) Show that $(e^z)' = e^z$. (Hint. Differentiate in the direction of the x-axis.)

(ii) Let f be any branch of $\log z$ (defined in an open set). Using the fact that f is inverse to e^z, show that f is holomorphic and $f'(z) = \frac{1}{z}$.

Exercise 2

Let γ be the sum of two line segments connecting -2 with iy and iy with 2, where y is a fixed parameter.

(i) Write an explicit parametrization for γ;

(ii) For every y, evaluate the integrals $\int_\gamma z\,dz$ and $\int_\gamma \bar{z}\,dz$. Which of the integrals is independent of y?

(iii) Use (ii) to show that the conclusion of Cauchy's theorem does not hold for $f(z) = \bar{z}$.

Exercise 3

(i) Calculate $\int_\gamma f(z)\,dz$, where $f(z) = \frac{1}{z}$ and $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, is the unit circle.

(ii) Use (i) to show that $f(z)$ does not have an antiderivative in its domain of definition.

(iii) Does $f(z) = \frac{1}{z^n}$ have an antiderivative, where $n \ge 2$ is an integer?

(iv) Give an example of a domain Ω, where the function $f(z) = \frac{1}{z^2(z-1)}$ does not have an antiderivative.

Justify your answer.

Exercise 4

Calculate the residues:

(i) $\mathrm{Res}_0 \frac{\sin(z^3) - e^z}{z^5 + e^z + z}$;

(ii) $\mathrm{Res}_1 \frac{\cos(2\pi z^2) + z}{e^z - e}$.

Exercise 5

Evaluate the integrals:

(i) $\int_{|z|=4} \frac{e^z}{(z^2-z)(z+5)}\, dz$;

(ii) $\int_0^{2\pi} \frac{\sin^2\theta}{3-\cos\theta} d\theta$;

(iii) $\int_{-\infty}^{\infty} \frac{x-x^2}{x^4-2x^2+2} dx$;

(iv) $\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^4-2x^2+2} dx,\ \lambda > 0$.