April 24, 2008 Lecturer Dmitri Zaitsev Hilary Term 2008

Course 414 2007-08

Due: after the lecture

Sheet 1

Exercise 1

Write $z = a + bi$, then $\mathsf{Im}(iz) = \mathsf{Im}(ai - b) = a = \mathsf{Re}z$, $\mathsf{Re}(iz) = \mathsf{Re}(ai - b) = -b = -\mathsf{Im}z$, $|\mathsf{Re}z| = |a| \le \sqrt{a^2 + b^2} = |z|$.

Exercise 2

Use formulas $\mathsf{log}z = \ln|z| + i\arg z$ and $\mathsf{Log}z = \ln|z| + i\mathrm{Arg}z$ with $\arg z$ being the set of all arguments and $-\pi < \mathrm{Arg}z \le \pi$ the principal value.

(i) $\mathsf{log}(i) = i(\pi/2 + 2\pi k)$, $k \in \mathbb{Z}$; $\mathsf{Log}(i) = i\pi/2$.

(ii) $\mathsf{log}(1+i) = \frac{\ln 2}{2} + i(\frac{\pi}{4} + 2\pi k)$, $k \in \mathbb{Z}$; $\mathsf{Log}(i) = \frac{\ln 2}{2} + \frac{i\pi}{4}$.

(iii) $\mathsf{log}\frac{2}{1-\sqrt{3}i} = -\mathsf{log}\frac{1-\sqrt{3}i}{2} = -\ln 2 - i(\mathsf{sin}^{-1}(-\sqrt{3}) + 2\pi k)$, $k \in \mathbb{Z}$; $\mathsf{Log}\frac{2}{1-\sqrt{3}i} = -\ln 2 + i(\mathsf{sin}^{-1}(\sqrt{3}))$.

Exercise 3

Write in polar coordinates $z_j = r_j e^{i\theta_j}$, $-\pi < \theta_j \le \pi$, $j = 1, 2$.

(i)

$$\arg(z_1 z_2) = \arg(r_1 r_2 e^{i(\theta_1+\theta_2)}) = \{\theta_1 + \theta_2 + 2\pi k : k \in \mathbb{Z}\},$$

$$\arg z_1 + \arg z_2 = \{\theta_1 + 2\pi k : k \in \mathbb{Z}\} + \{\theta_2 + 2\pi l : l \in \mathbb{Z}\} = \{\theta_1 + \theta_2 + 2\pi(k+l) : k, l \in \mathbb{Z}\}.$$

Clearly both sets coincide.

(ii) Since $-\pi < \theta_1 + \theta_2 \le \pi$, we have $\mathrm{Arg}(z_1 z_2) = \theta_1 + \theta_2 = \mathrm{Arg}z_1 + \mathrm{Arg}z_2$.

(iii) Take $z_1 = z_2 = -i$, then $\mathrm{Arg}(z_1 z_2) = \pi$ but $\mathrm{Arg}z_1 + \mathrm{Arg}z_2 = -\pi$.

Sheet 2

Exercise 1

(i) Annulus with center 0, inner radius 1 and outer radius 2.

(ii) Annulus with center $-i$, inner radius 1 and outer radius 2.

(iii) Half-plane $y \le x - 2$.

Exercise 2

We have

$$\frac{\bar{f}(z) - \bar{f}(z_0)}{z - z_0} = \frac{\overline{f(\bar{z})} - \overline{f(\bar{z})}}{z - z_0} = \overline{\left(\frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0}\right)}$$

and the last expression has limit as $z \to z_0$ for any $z_0 \in \bar{\Omega}$.

Exercise 3

(i) Writing $f = u + iv$ with u, v real, we have $u = \text{const}$, hence $u_x = u_y = 0$. By the Cauchy-Riemann equations, $v_x = -u_y = 0$, $v_y = u_x = 0$, hence also $v = \text{const}$ as desired.

(ii) Dividing by a and taking real part, we have

$$u + \mathsf{Re}\left(\frac{b}{a}\right) v = \mathsf{Re}\left(\left(1 - i\mathsf{Re}\frac{b}{a}\right) f\right) = \text{const}.$$

Then by part (i), $\left(\left(1 - i\mathsf{Re}\frac{b}{a}\right) f\right) = \text{const}$ and hence $f = \text{const}$.

Exercise 4

(i) $\Omega = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ with a branch given by $f(z) = \sqrt{|z|}e^{i\text{Arg}z/3}$. Any other branch is $\tau f(z)$, where τ is any 3rd root of unity. Since the limits of $\tau f(z)$ for every τ from above and below do not coincide at every $z \in \mathbb{R}_{>0}$, τf cannot be continuously extended to any such point. If there were a larger open set $\widetilde{\Omega}$ with a branch $\widetilde{f}$, that branch would be a holomorphic extension of some branch τf. Hence $\widetilde{\Omega}$ cannot contain any point from $\mathbb{R}_{>0}$ and since it is open, it also cannot contain 0. Thus Ω is maximal.

(ii) $\Omega = \mathbb{C} \setminus \mathbb{R}_{\geq -1}$ with the branches given by $f_k(z) = \ln|z| + i\text{Arg}(z+1) + 2i\pi k$, $k \in \mathbb{Z}$. Maximality of Ω is shown by the same argument as in (i).

(iii) $\Omega = \mathbb{C}$ with the branches $\pm e^{\frac{z}{2}}$.

Exercise 5

Define $\widetilde{\Phi}\colon [0,1] \times [a,b] \to \Omega$ by

$$\widetilde{\Phi}(t,\lambda) := \begin{cases} \Phi(\frac{3(\lambda-a)}{b-a}t, a), & a \leq \lambda \leq \frac{2a+b}{3}, \\ \Phi(t, 3\lambda - a - b), & \frac{2a+b}{3} \leq \lambda \leq \frac{a+2b}{3}, \\ \Phi(\frac{3(b-\lambda)}{b-a}t, a), & \frac{a+2b}{3} \leq \lambda \leq b, \end{cases}$$

then $\widetilde{\Phi}$ the homotopy with fixed endpoints. Moreover, $\int_{\gamma_t} f dz = \int_{\widetilde{\gamma}_t} f dz$ with $\widetilde{\gamma}_t(\lambda) := \widetilde{\Phi}(t,\lambda)$. The conclusion follows from Cauchy's theorem.

Sheet 3

Exercise 1

(i)

$$\gamma(t) = \begin{cases} t + iy(1+t), & -1 \leq t \leq 0 \\ t + iy(1-t), & 0 \leq t \leq 1 \end{cases}$$

(ii)

$$\int_\gamma z\, dz = \int_{-1}^0 (t + iy(1+t))(1+iy)dt + \int_{-1}^0 (t + iy(1-t))(1-iy)dt = \ldots$$

$$\int_\gamma \bar{z}\, dz = \int_{-1}^{0} (t - iy(1+t))(1+iy)dt + \int_{-1}^{0} (t - iy(1-t))(1-iy)dt = \dots$$

Exercise 2 (i) For every arc $\gamma\colon [0, 2\pi] \to \mathbb{C}$ with $\gamma(0) = \gamma(2\pi)$, $\partial[\gamma] = 0$, hence $[\gamma]$ is a cycle and sums of cycles are cycles.

(ii) $[\gamma_4] - [\gamma_r]$ bounds an annulus inside Ω that can be triangulated into a sum of 2-chains. Hence $[\gamma_4] - [\gamma_r]$ is the boundary of a 2-chain and hence is null-homologous for each r. Then also the sum

$$([\gamma_4] - [\gamma_2]) + ([\gamma_4] - [\gamma_3]) = 2[\gamma_4] - ([\gamma_2] + [\gamma_3])$$

is null-homologous and the needed conclusion follows.

(iii) γ_2 and γ_3 are homotopic to each other but are not homotopic to λ in Ω. The cycles $[\gamma_2]$ and $[\gamma_3]$ are homologous but not homologous to $[\lambda]$ in Ω. A homotopy between γ_2 and γ_3 is given by $\Phi(s, t) = (2 + s)e^{it}$, $0 \le s \le 1$. The cycle $[\gamma_2] - [\gamma_3]$ bounds an annulus in Ω that can be triangulated into a sum of 2-chains, hence it is null-homologous. The integral of $f(z) = \frac{1}{z}$ over λ is 0 by the Cauchy's theorem. On the other hand, the integral of $f(z)$ over γ_r is $2i\pi$. This proves the claims.

If Ω is replaced by $\mathbb{C}$, all arcs become homotopic and all cycles homologous.

Exercise 3

(i) The equation $z^2 - iz + 2 = 0$ has solutions $z = -i$ and $z = 2i$. Hence the maximal disk cetered at 0, where the function is holomorphic, has radius 1 and therefore the radius of convergence is 1.

(ii) The radius is again 1 with the same argument.

Exercise 4

(i) Convergence is uniform on every compactum in $\mathbb{C}$, hence the maximal open set is $\mathbb{C}$.

(ii) Convergence is uniform on every compactum in the unit disk Δ. The sequence is divergent at every z with $|z| > 1$. Thus the unit disk cannot be replaced by any larger open set and is hence maximal.

(iii) The sequence is convergent for every z with $\mathsf{Re}z < 0$ and divergent for every z with $\mathsf{Re}z > 0$. Hence the desired open set Ω is contained in $\mathsf{Re}z < 0$. Direct calculation shows that the sequence converges uniformly on every set $\mathsf{Re}z \le -\varepsilon$ for $\varepsilon > 0$, hence on every compactum in Ω. The divergence for $\mathsf{Re}z > 0$ shows that Ω is maximal.

Exercise 5

(i) If (f_n) and (g_n) converge uniformly to functions f and g on a compactum K, then

$$\sup_{z\in K} |f_n(z) - f(z)| \to 0, \quad \sup_{z\in K} |g_n(z) - g(z)| \to 0,$$

as $n \to \infty$, which implies

$$\sup_{z \in K} |(f_n(z) + g_n(z)) - (f(z) + g(z))| \to 0,$$

proving uniform convergence of $f_n + g_n$ to $f + g$ on K. The corresponding proof for $f_n g_n$ follows from the estimate

$$\begin{aligned} |f_n(z)g_n(z) - f(z)g(z)| &= |f_n(z)g_n(z) - f_n(z)g(z) + f_n(z)g(z) - f(z)g(z)| \\ &\le |f_n(z)||g_n(z) - g(z)| + |g(z)||f_n(z) - f(z)|, \end{aligned}$$

the boundedness of g on K and uniform boundedness of f_n on K.

(ii) The sequence f_n/g_n is not always convergent, e.g. take constant functions $f_n = 1$, $g_n = 1/n$.

Exercise 6

(i) The function is holomorphic away from its poles $z = \pm i$. Hence it is holomorphic in two maximal rings centered at i:

$$R_1 := \{0 < |z - i| < 2\}, \quad R_2 := \{2 < |z - i|\}$$

Expand into powers of $(z - i)$:

$$f(z) = \frac{1}{(z+i)(z-i)} = \frac{1}{z-i}\frac{1}{2i + (z-i)}$$

In R_1 we have the Larent series

$$f(z) = \frac{1}{z-i}\frac{1}{2i(1 + (z-i)/2i)} = \frac{1}{2i}\frac{1}{z-i}\sum_{k=0}^{\infty}\left(-\frac{z-i}{2i}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k(z-i)^{k-1}}{(2i)^{k+1}},$$

whose ring of convergence is R_1.

In R_2 we have the Larent series

$$f(z) = \frac{1}{(z-i)^2}\frac{1}{(1 + 2i/(z-i))} = \frac{1}{(z-i)^2}\sum_{k=0}^{\infty}\left(\frac{-2i}{z-i}\right)^k = \sum_{k=0}^{\infty}\frac{(-2i)^k}{(z-i)^{k+2}}$$

whose ring of convergence is R_2.

(ii) The function is holomorphic in the ring $R := \{0 < |z - 1| < 1\}$ centered at 1, which is maximal with this property. Hence there is one Laurent series expansion with ring of convergence R. To find it, expand into powers of $(z - 1)$:

$$f(z) = \frac{1}{(z-1)^2}\mathsf{Log}(1+(z-1)) = \frac{1}{(z-1)^2}\sum_{k=0}^{\infty}\frac{(-(z-1))^k}{k} = \sum_{k=0}^{\infty}\frac{(-(z-1))^{k-2}}{k}.$$

(iii) The function is holomorphic away from its poles $z = 0$ and $z = 2$, hence it is holomorphic in two maximal rings centered at 2:

$$R_1 := \{0 < |z-2| < 2\}, \quad R_2 := \{2 < |z-2|\}.$$

The corresponding Laurent series with rings of convergence R_1 and R_2 are respectively

$$\frac{\mathsf{cos}(\pi(z-2))}{(z-2)^3(2+(z-2))} = \frac{1}{(z-2)^3}\frac{1}{2(1+(z-2)/2)}\sum_{k=0}^{\infty}\frac{(-\pi(z-2))^k}{k!}$$

$$= \frac{1}{(z-2)^3}\frac{1}{2}\left(\sum_{s=0}^{\infty}\left(-\frac{z-2}{2}\right)^s\right)\left(\sum_{k=0}^{\infty}\frac{(-\pi(z-2))^k}{k!}\right) = \sum_{s,k\geq 0}\frac{(-\pi)^k(-1)^s}{2^{s+1}k!}(z-2)^{s+k-3}$$

$$= \sum_{l=-3}^{\infty}\left(\sum_{s,k\geq 0, s+k-3=l}\frac{(-\pi)^k(-1)^s}{2^{s+1}k!}\right)(z-2)^l$$

and

$$\frac{\mathsf{cos}(\pi(z-2))}{(z-2)^3(2+(z-2))} = \frac{1}{(z-2)^4}\frac{1}{1+2/(z-2)}\sum_{k=0}^{\infty}\frac{(-\pi(z-2))^k}{k!}$$

$$= \frac{1}{(z-2)^4}\left(\sum_{s=0}^{\infty}\left(-\frac{2}{z-2}\right)^s\right)\left(\sum_{k=0}^{\infty}\frac{(-\pi(z-2))^k}{k!}\right) = \sum_{s,k\geq 0}\frac{(-\pi)^k(-2)^s}{k!}(z-2)^{k-s-4}$$

$$= \sum_{l=-\infty}^{\infty}\left(\sum_{s,k\geq 0, k-s-4=l}\frac{(-\pi)^k(-2)^s}{k!}\right)(z-2)^l.$$

Exercise 7

The formula for the Laurent series expansion of the product fg is obtained by taking the product of both Laurent series:

$$f(z)g(z) = \sum_{k,n}a_k b_n(z-z_0)^{k+n} = \sum_{l}\left(\sum_{k,n,k+n=l}a_k b_n\right)(z-z_0)^l = \sum_{l}c_l(z-z_0)^l.$$

To prove convergence, remark that the Laurent series converge absolutely in their rings of convergence, hence we have

$$\sum_n |a_n||z-z_0|^n < \infty, \quad \sum_k |b_k||z-z_0|^k < \infty$$

and then

$$\sum_{k,n} |a_k b_n||z-z_0|^{k+n} = \sum_l \left(\sum_{k,n,k+n=l} |a_k b_n| \right) |z-z_0|^l < \infty.$$

In particular, each coefficient $c_l = \sum_{k,n,k+n=l} a_k b_n$ is well-defined (given by absolutely convergent series), and the Laurent series $\sum_l c_l (z-z_0)^l$ is convergent in the given ring.

Exercise 8

Use the Taylor series expansion in powers of $(z-z_0)$:

(i) Write

$$e^{z\mathsf{cos}z - z} = e^{z(1-z^2/2+\dots)-z} = e^{z^3/2+\dots}$$

$$= 1 + (z^3/2+\dots) + (z^3/2+\dots)^2/2 + \dots = 1 + z^3/2 + \dots,$$

then the multiplicity at 0 is the vanishing order of $f - f(0)$, which is 3.

(ii) Write

$$(\mathsf{Log}(\mathsf{cos}z))^2 = (\mathsf{Log}(\mathsf{cos}(z-2\pi)))^2 = (\mathsf{Log}(1-(z-2\pi)^2/2+\dots))^2$$

$$= (1-((z-2\pi)^2/2+\dots)+\dots)^2 = 1-(z-2\pi)^2+\dots,$$

then the multiplicity at 2π is the vanishing order of $f - f(2\pi)$, which is 2.

(iii) Write

$$(1+z^2-e^{z^2})^4 = (1+z^2-(1+z^2+z^4/2+\dots))^4 = (-z^4/2+\dots)^4 = z^{16}/2^4+\dots,$$

hence the multiplicty is 16.

Sheet 4

Exercise 1

(i) The function is ratio of two holomorphic functions, hence it is meromorphic and therefore the singularity is either removable or pole. Expanding in the powers of $(z-z_0)$, we obtain

$$\frac{\mathsf{sin}z}{z-\pi} = \frac{-\mathsf{sin}(z-\pi)}{z-\pi} = \frac{-(z-\pi)+\dots}{z-\pi} = -1+\dots,$$

which is a Laurent series with trivial principal part, hence the singularity is removable. The residue is zero.

(ii) The function is again meromorphic and we have

$$\frac{z}{\cos z - 1} = \frac{z}{(1 - z^2/2 + \ldots) - 1} = \frac{-1}{z/2 + \ldots} = \frac{-2}{z}(1 + \ldots),$$

hence we have a pole at 0 and the Laurent coefficient of $1/z$ is -2. Thus the residue is -2.

(iii) Expand as before:

$$ze^{-1/z^3} = z \sum_{k=0}^{\infty} \left(\frac{-1}{z^3}\right)^k \frac{1}{k!},$$

which is a Laurent series with infinite principal part. Hence the singularity is essential and the residue is the coefficient of $1/z$, which is 0.

(iv) The function has poles when $e^{1/z} = 1$, i.e. at the points $z = \frac{1}{2i\pi k}$, $k \in \mathbb{Z}$. Hence 0 is not an isolated singularity.

Exercise 2

$\Omega = \mathbb{C}$ for (i) and (ii) and $\Omega = \mathbb{C} \setminus \{0\}$ for (iii) and (iv).

Exercise 3

(i) $f + g$ is always defined and holomorphic in a punctured disk centered at z_0, hence has an isolated singularity there.

(ii) $f + g$ may not have a pole, e.g. $f(z) = 1/z$, $g(z) = -1/z$.

(iii) fg always has isolated singularity and a pole, which follows from the factorization lemma into a power of $(z - z_0)$ and a nonvanishing holomorphic function.

(iv) the pole order of fg is the sum of pole orders of f and g as follows from the factorization. The pole order of $f+g$ is best understood by looking at the Laurent series expansions. If the pole orders of f and g are different, the term with the lowest power of $(z - z_0)$ is present in the expansion of $f + g$, hence in this case the pole order of $f + g$ is the maximum of the pole orders of f and g. On the other hand, if both orders are equal, there may be some cancellation of terms when adding two Laurent series expansions. In that case the pole order of $f + g$ can be any number between 1 and the maximum of the pole orders of f and g.

Exercise 4

If $g = 0$, then $f = 0$ and the needed conclusion is trivial. Otherwise f/g is meromorphic, in particular all its singularities are isolated. Since $|f/g| \leq 1$ away from the singularities, the Riemann extension theorem implies that all singularities are removable. Hence $h = f/g$ extends to an entire function, which is bounded by 1 aways from singularities and hence everywhere by continuity. Now Liouville's theorem implies that h is constant, hence the conclusion.

Exercise 5

Since f is bounded outside the disk, its poles can only be inside, hence there are finitely many of them (the set of poles has no limit points in the set where f is meromorphic).

We prove the claim by induction on the number of poles. If there are no poles, f is holomorphic and bounded both inside and outside the disk. Hence Liouville's theorem applies and f must be constant, hence rational.

Now suppose the claim holds whenever the number of poles is $< k$ and consider f with k poles. Let z_0 be one of them. Expand f in Laurent series near z_0:

$$f(z) = \sum_{k<0} c_k(z - z_0)^k + \sum_{k\geq 0} c_k(z - z_0)^k = P(z) + R(z),$$

where the principal part $P(z)$ is a finite sum, hence rational and bounded outside the given disk $B_R(0)$. Moreover, the only pole of P is at z_0. Therefore, $R(z) := f(z) - P(z)$ is also meromorphic on $\mathbb{C}$, bounded outside the disk $B_R(0)$ and has less poles than $f(z)$. By the induction assumption, $R(z)$ is rational. Since $P(z)$ is rational, $f(z) = P(z)+R(z)$ is rational as desired.

Exercise 6

(i) Set $F(z) = 5z^4$, $f(z) = z^6 + z^3 - 2z$, then $|f(z)| < |F(z)|$ on the unit circle, hence by Rouché's theorem, the number of zeroes inside the circle for $F + f$ is the same as for F, which is 4.

(ii) Similarly setting $F(z) = 7$ and $f(z) = 2z^4 - 2z^3 + z^2 - z$, we see that the number of zeroes of $F + f$ inside the unit circle is 0.

Exercise 7

(i) $\varphi(z) = \frac{z-1}{z}\frac{1}{2}$;

(ii) $\varphi(z) = \frac{z-1}{z-a}a$ for any $a \notin \{0, 1\}$ or $\varphi(z) = 1 - z$ (which is corresponds to $a = \infty$);

(iii) Substituting $\varphi(0) = \infty$ into $\varphi(z) = \frac{az+b}{cz+d}$ gives $d = 0$, thus $\varphi(z) = \frac{az+b}{z}$ (dividing by c and renaming a and b) for any $a \neq 0$ and any b.

Exercise 8

If $z_0 \in \Omega$ is a limit point of $f^{-1}(\infty)$, then $f(z_0) = \infty$ and hence $g := 1/f$ is holomorphic in a neighborhood of z_0, with z_0 being a limit point for the set of zeroes of g. Then, by the identity principle, $g \equiv 0$ and hence $f \equiv \infty$ in a neighborhood of z_0.

Assume that the set $S \subset \Omega$ consisting of all limit points of $f^{-1}(\infty)$ is nonempty. It is clearly a closed set in Ω. Furthermore, the above remark implies that S is also open in Ω. Since Ω is connected, it follows that $S = \Omega$ and therefore $f \equiv \infty$.

Exercise 9

The function e^z maps the strip $0 < \mathsf{Im} z < 2\pi$ biholomorphically onto the upper half-plane, hence $f(z) := e^{\pi(z+i)}$ maps $|\mathsf{Im} z| < 1$ biholomorphically onto the upper half-plane with the inverse $f^{-1}(w) = \frac{\mathsf{Log} w}{\pi} - i$. Any biholomorphic automorphism of the upper half-plane is a Möbius transformation $\varphi(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$. Hence any biholomorphic automorphism of $|\mathsf{Im} z| < 1$ is of the form $\widetilde{\varphi} = f^{-1} \circ \varphi \circ f$ with f and φ as above.

Exercise 10

Any 3 disjoint points of $\mathbb{C}$ either belong to one uniquely determined line or a circle (use elementary geometry). If any of these points is ∞, a suitable Möbius transformation φ maps them all into $\mathbb{C}$, where they determine an unique generalized circle C, then $\varphi^{-1}(C)$ is the uniquely determined generalized circle through the given points.

Now suppose we are given two generalized circles C_1 and C_2. Take any 3 disjoint points on each of them. Then there exists a Möbius transformation ψ sending the first triplet of points into the second. It always sends generalized circles into generalized circles. Furthermore, the uniqueness of such circles through a triplet of points implies that $\psi(C_1) = C_2$.