Course 2325 2010 Complex Analysis I

S h e e t 3

Due: at the end of the lecture

Exercise 1

(i) Show that $(e^z)' = e^z$. (Hint. Differentiate in the direction of the x-axis.)

Solution. Since

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} =$$

$$= \lim_{h \in \mathbb{R},\, h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \partial_x f(z_0)$$

holds for any holomorphic $f(z)$, we have

$$(e^z)' = \partial_x e^x(\cos y + i \sin y) = e^x(\cos y + i \sin y) = e^z.$$

Exercise 2

(iii) Use (ii) to show that the conclusion of Cauchy's theorem does not hold for $f(z) = \bar{z}$.

Solution. It follows from (ii) that

$$\int_{\gamma_1} \bar{z}\, dz \neq \int_{\gamma_2} \bar{z}\, dz$$

whenever γ_j corresponds to some y_j with $y_1 \neq y_2$. Form a closed path C by following γ_1 from -1 to 1 and then γ_2 with reversed orientation from 1 to -1. Then

$$\int_C \bar{z}\, dz = \int_{\gamma_1} \bar{z}\, dz - \int_{\gamma_2} \bar{z}\, dz \neq 0.$$

Thus we have a closed path for which the conclusion of Cauchy's theorem does not hold with $f(z) = \bar{z}$.

Exercise 3

(i) Calculate $\int_\gamma f(z)\,dz$, where $f(z) = \frac{1}{z}$ and $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, is the unit circle.

Solution.

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{ie^{it}\,dz}{e^{it}} = 2\pi i.$$

(ii) Use (i) to show that $f(z)$ does not have an antiderivative in its domain of definition.

Solution. It f had antiderivative, its integral would be zero over any closed path, which is not the case.

(iii) Use Exercise 1 (ii) to give an example of a domain Ω, where f does have an antiderivative.

Solution. Take $\Omega = \mathbb{C} \setminus \{0\}$, the domain of definition of $f(z) = \frac{1}{z}$, then (ii) shows that f does not have antiderivative in Ω.

(iv) Does $f(z) = \frac{1}{z^n}$ have an antiderivative, where $n \ge 2$ is an integer?

Solution. Yes, take $F(z) = \frac{1}{(-n+1)z^{n-1}}$.

Justify your answer.