Course 2325 2010 Complex Analysis I

S h e e t 2

Due: at the end of the lecture

Exercise 1

Consider the sequence of holomorphic functions $f_n(z) = z + \frac{1}{n}$.

(i) Is the sequence (f_n) converging uniformly on $\mathbb{C}$?

(ii) Is the sequence of squares (f_n^2) converging uniformly on $\mathbb{C}$?

Justisfy your answer.

Solution

(i) The sequence (f_n) has pointwise limit $f(z) = z$. Then the convergence is uniform on $\mathbb{C}$ if and only if

$$\sup_{z\in\mathbb{C}} |f_n(z) - f(z)| \to 0, \quad n \to \infty.$$

In our case $\sup_{z\in\mathbb{C}} |f_n(z) - f(z)| = \frac{1}{n}$, hence the convergence is uniform.

(ii) The sequence of squares (f_n^2) converges pointwise to the square f^2. However, this time

$$\sup_{z\in\mathbb{C}} |f_n(z)^2 - f(z)^2| = \sup_{z\in\mathbb{C}} |\frac{2z}{n} + \frac{1}{n^2}| = +\infty,$$

hence the left-hand side does not converge to 0 and therefore the convergence is not uniform.

Exercise 2

Use the Cauchy-Riemann equations to decide which of the following functions are holomorphic:

$$\mathsf{Im} z, \quad -i|z|^2, \quad \bar{z}^2, \quad e^{z+i}, \quad e^{\bar{z}}.$$

Solution We use the complex form of the Cauchy-Riemann equation: $\partial_{\bar{z}} f = 0$, where $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. Then we obtain

$$2\partial_{\bar{z}}(\mathsf{Im} z) = (\partial_x + i\partial_y)y = i \neq 0,$$

$$2\partial_{\bar{z}}(-i|z^2|) = (\partial_x + i\partial_y)(-ix^2 - iy^2) = -2ix + 2y \neq 0,$$

$$2\partial_{\bar{z}}(\bar{z}^2) = (\partial_x + i\partial_y)(x^2 - 2ixy - y^2) = 2x - 2iy + 2x - 2iy \neq 0$$

$$2\partial_{\bar{z}}(e^{z+i}) = (\partial_x + i\partial_y)(e^x(\mathsf{cos}(y+1) + i\mathsf{sin}(y+1))) =$$

$$= e^x(\mathsf{cos}(y+1) + i\mathsf{sin}(y+1)) + ie^x(-\mathsf{sin}(y+1) + i\mathsf{cos}(y+1) = 0,$$

$$2\partial_{\bar{z}}(e^{\bar{z}}) = (\partial_x + i\partial_y)(e^x(\mathsf{cos}y - i\mathsf{sin}y)) =$$

$$= e^x(\mathsf{cos}y - i\mathsf{sin}y) + ie^x(-\mathsf{sin}y - i\mathsf{cos}y) \neq 0,$$

thus only e^{z+i} is holomorphic.