November 1, 2010 Lecturer Dmitri Zaitsev Michaelmas Term 2010

Course 2325 2010 Complex Analysis I

S h e e t 1

Due: at the end of the lecture

Exercise 1

Find $\log z$, $\mathrm{Log} z$ and $\sqrt{z}$ for

(i) $z = 2i$;

(ii) $z = 1 - i$;

(iii) $z = 2/(1+\sqrt{3}i)$.

Solution

Use formulas $\log z = \log|z| + i \arg z$ and $\mathrm{Log} z = \log|z| + i\mathrm{Arg} z$ with $\arg z$ being the set of all arguments and $-\pi < \mathrm{Arg} z \leq \pi$ the principal value.

(i) $\log z = \log|z| + i\arg(z) = \log 2 + i\pi/2 + \{2\pi k : k \in \mathbb{Z}\}$,
$\mathrm{Log} z = \log 2 + i\pi/2$,
$\sqrt{z} = \pm\sqrt{2}e^{i\pi/4}$.

(ii) $\log z = \log\sqrt{2} - i\pi/4 + \{2\pi k : k \in \mathbb{Z}\}$,
$\mathrm{Log} z = \log\sqrt{2} - i\pi/4$,
$\sqrt{z} = \pm 2^{1/4}e^{-i\pi/8}$.

(iii) $z = 2/(2e^{i\pi/3}) = e^{-i\pi/3}$
$\log z = -i\pi/3 + \{2\pi k : k \in \mathbb{Z}\}$,
$\mathrm{Log} z = -i\pi/3$'
$\sqrt{z} = \pm e^{-i\pi/6}$.

Exercise 2

Prove that $\mathrm{Im}(iz) = \mathrm{Re} z$, $\mathrm{Re}(iz) = -\mathrm{Im} z$, $e^{\overline{z}} = \overline{e^z}$, $e^{-z} = \frac{1}{e^z}$.

Solution

Let $z = a + ib$, then $iz = -b + ia$, so clearly $\mathrm{Im}(iz) = a = \mathrm{Re} z$ and $\mathrm{Re}(iz) = -b = -\mathrm{Im} z$.

$\overline{z} = a - ib$. So, $e^{\overline{z}} = e^a e^{-ib} = e^a(\cos(-b) + i\sin(-b)) = e^a(\cos b - i\sin b)$, whereas $e^z = e^a(\cos b + i\sin b)$, so $e^{\overline{z}} = \overline{e^z}$.

$$\frac{1}{e^z} = \frac{1}{e^a(\cos b + i\sin b)} = \frac{1}{e^a(\cos b + i\sin b)}\frac{\cos b - i\sin b}{\cos b - i\sin b} = e^{-a}\frac{\cos b - i\sin b}{\cos^2 b + \sin^2 b} = e^{-a}e^{-ib} = e^{-z}$$

Exercise 3

(i) Show that $\mathsf{log}(z_1 z_2) = \mathsf{log} z_1 + \mathsf{log} z_2$ as sets.

(ii) Show that $\mathsf{Log}(z_1 z_2) = \mathsf{Log} z_1 + \mathsf{Log} z_2$ provided $-\pi < \text{Arg} z_1 + \text{Arg} z_2 < \pi$

(iii) Give an example of z_1, z_2 with $\mathsf{Log}(z_1 z_2) \neq \mathsf{Log} z_1 + \mathsf{Log} z_2$.

Solution

(i) Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$ and hence $\mathsf{log}(z_1 z_2) = \mathsf{log} r_1 r_2 + i(\theta_1 + \theta_2) + i\{2\pi k : k \in \mathbb{Z}\} = \mathsf{log} r_1 + i\theta_1 + \mathsf{log} r_2 + i\theta_2 + i\{2\pi k : k \in \mathbb{Z}\} = \mathsf{log} z_1 + \mathsf{log} z_2$

(ii) We know from (i) that $\mathsf{log}(z_1 z_2) = \mathsf{log} z_1 + \mathsf{log} z_2$ as sets, hence $\arg(z_1 z_2) = \{\theta_1 + \theta_2 + 2\pi k : k \in \mathbb{Z}\}$, but $-\pi < \text{Arg}(z_1 z_2) < \pi$, so $\text{Arg}(z_1 z_2) = \text{Arg} z_1 + \text{Arg} z_2$ only if $-\pi < \text{Arg} z_1 + \text{Arg} z_2 < \pi$. So, $\mathsf{Log}(z_1 z_2) = \mathsf{Log} z_1 + \mathsf{Log} z_2$ only if $-\pi < \text{Arg} z_1 + \text{Arg} z_2 < \pi$.

(iii) Let $z_1 = z_2 = e^{i2\pi/3}$. Then $\theta_1 + \theta_2 = 4\pi/3 > \pi$. $\mathsf{Log} z_1 + \mathsf{Log} z_2 = i4\pi/3$, but $\mathsf{Log}(z_1 z_2) = -i2\pi/3$.

Exercise 4

Using the definition show that:

(i) Finite intersections and arbitrary unions of open sets are open.

(ii) Finite unions and arbitrary intersections of closed sets are closed.

Solution

(i) By the definition of an open set, for each point z in an open set U there exists an $\varepsilon > 0$ such that the open ball $B_\varepsilon(z) = \{\xi : |\xi - z| < \varepsilon\}$ is contained in the set U.

If now $\bigcup_\alpha U_\alpha$ is an arbitrary union, then $z \in \bigcup_\alpha U_\alpha$, then $z \in U_i$ for some i, so there is an $\varepsilon > 0$ such that $B_\varepsilon(z) \subset U_i$, and hence $B_\varepsilon(z) \subset \bigcup_\alpha U_\alpha$.

If on the other hand, $U_1 \cap \cdots \cap U_n$ is a finite intersection and $z \in U_1 \cap \cdots \cap U_n$ for U_j open, then there are ε_j such that $B_{\varepsilon_j}(z) \subset U_j$ for every $j = 1, \ldots, n$. Choose $\varepsilon = \min\{\varepsilon_j : 1 \leq j \leq n\}$, then $B_\varepsilon(z) \subset U_1 \cap \cdots \cap U_n$, so finite intersections of open sets are open.

(ii) Since we defined a closed set to be the complement of an open set, it suffices to note that $\mathbb{C} \backslash (\bigcup_\alpha U_\alpha) = \bigcap_\alpha (\mathbb{C} \backslash U_\alpha)$ and that $\mathbb{C} \backslash (U_1 \cap \ldots \cap U_n) = (\mathbb{C} \backslash U_1) \cup \ldots \cup (\mathbb{C} \backslash U_n)$ and the problem is reduced to (i).

Exercise 5

Construct a branch of $\mathsf{log} z$ on the set $\mathbb{C} \backslash \{-iy : y \geq 0\}$. Show that the branch you constructed is indeed continuous.

Solution Since we know what $\mathsf{log}z$ is for polar coordinates, we should redefine the set on which we want to define the branch:

$$\mathbb{C}\backslash\{iy : y \geq 0\} = \{re^{i\theta} : r > 0, -\pi/2 < \theta < 3\pi/2\}$$

Now, we define our branch of $\mathsf{log}z$ to be:

$$\mathsf{log}z = \begin{cases} r + i(\text{Arg}z + 2\pi) & \text{if } -\pi < \text{Arg}z < -\pi/2; \\ r + i\text{Arg}z & \text{if } -\pi/2 < \text{Arg}z \leq \pi. \end{cases}$$

Now we need to show that this branch is continuous. The only points where continuity is an issue is at the line $\{re^{i\theta} : \theta = \pm\pi\}$. At the line the branch takes the value $r + i\pi$, and taking the limit as $\theta \to -\pi$ the branch clearly tends towards $r + i\pi$ too.

Exercise 6

Sketch the set of points given by the condition:

(i) $1 < |z| < 2$,

(ii) $1 < |z + 2i| < 2$,

(iii) $\mathsf{Re}((1 + i)\overline{z}) \geq 4$.

Solution Parts (i) & (ii) are both annuli of inner radius 1 and outer radius 2, centered at 0 and $-2i$ respectively. If $z = a + ib$ then $\mathsf{Re}((1 + i)\overline{z}) = a + b$, so part (iii) is to the right of the diagonal line $a + b = 4$.

Exercise 7

Find all z for which the following identity holds:

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

Solution $\frac{1}{4z-z^2} = \frac{1}{z}\frac{1}{4-z}$. Now, $\frac{1}{4-z} = \frac{1}{4}\frac{1}{1-z/4} = \frac{1}{4}(\sum_{n=0}^{\infty}(\frac{z}{4})^n)$, whenever $|z/4| < 1$, or in other words $|z| < 4$. Since $\frac{1}{4z-z^2}$ is not defined for $z = 0$ and the sum diverges for $|z| \geq 4$, we get that the identity holds for $0 < |z| < 4$.