COMPLEX ANALYSIS: LECTURE NOTES

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Contents

1. The origin of complex numbers

1.1. **Solving quadratic equation.** The simplest quadratic equation having no real solutions is

$$x^2 + 1 = 0.$$

Trying to solve it, we have to introduce the symbol $i = \sqrt{-1}$. As we also want to add and multiply, we are lead to consider *formal expressions*

$$z = a + ib, \quad a, b \in \mathbb{R},$$

where a and b are arbitrary *real numbers*. We call z a *complex number* and write

$$a = \operatorname{Re} z, \quad b = \operatorname{Im} z$$

for the *real* and *imaginary* parts of z.

More formally, the expression $a + ib$ is a way to write the pair (a, b), that is more convenient to define algebraic operations than using pairs.

Introducing i may appear artificial and is not relevant to the problem of finding *real* solutions. In fact, solution of quadratic equation was known to the ancient Greeks, who never came across complex numbers nor felt any need for it.

1.2. **Cubic equation and Cardano's formula.** In contrast to quadratic equations, solving a cubic equation even over reals forces you to pass through complex numbers. In fact, this is how complex numbers were discovered.

Consider the general cubic equation

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \quad a_3 \neq 0.$$

Dividing by a_3 we can reduce it to

$$x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3} = 0.$$

Using binomial formula $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$, we can eliminate the term with x^2 by rewriting it as

$$(x + \frac{a_2}{3a_3})^3 + p(x + \frac{a_2}{3a_3}) + q = 0$$

for suitable p and q, or after the change of variable $y = x + \frac{a_2}{3a_3}$,

$$y^3 + py + q = 0. \tag{1.1}$$

Cardano's formula for solving (1.1) is

$$y = u+v, \quad u = \sqrt[3]{-\frac{q}{2}+\sqrt{D}}, \quad v = \sqrt[3]{-\frac{q}{2}-\sqrt{D}}, \quad D = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3,$$

such that u, v satisfy

$$uv = -\frac{p}{3}. \tag{1.2}$$

Condition (1.2) is sometimes forgotten when Cardano's formula is stated, but is essential and cannot be dropped, or else the formula gives other values of y that are not among solutions of (1.1).

The main difference from the quadratic equation formula is the fact that, in Cardano's formula, we may need to use in our calculations *complex numbers* even when the final result is *real*. We illustrate this by a simple example.

1.3. **Example of using Cardano's formula.** Consider the cubic equation

$$x^3 - 3x = 0.$$

Since $x^3 - 3x = x(x-\sqrt{3})(x+\sqrt{3})$, we have 3 real solutions

$$x = 0, \pm\sqrt{3}.$$

Now in Cardano's formula we have $p = -3$, $q = 0$, $D = -1$, and hence

$$u = \sqrt[3]{\sqrt{-1}}, \quad v = \sqrt[3]{-\sqrt{-1}}.$$

Thus, even if all solutions are real, we need to evaluate $\sqrt{-1}$ and then take cubic roots! We shall now introduce the necessary tools to complete the calculation.

2. Algebraic operations for complex numbers

2.1. **Addition and multiplication.** Using the rule $i^2 = -1$ we can define *addition* and *multiplication* of complex numbers:

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2),$$

$$(a_1+ib_1)(a_2+ib_2) = (a_1a_2)+i(a_1b_2+a_2b_1)+i^2(b_1b_2) = (a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1).$$

Both addition and multiplication are *commutative* and *associative*

(1) commutativity: $z_1 + z_2 = z_2 + z_1$, $z_1z_2 = z_2z_1$;
(2) associativity: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, $(z_1z_2)z_3 = z_1(z_2z_3)$.

Both addition and multiplication have the identity elements - 0 and 1 respectively, i.e.

(1) additive identity: $z + 0 = 0 + z = z$;

(2) multiplicative identity: $1z = z1 = z$.

Every $z = a + ib$ has the *additive inverse*

$$-z = (-a) + i(-b) \implies z + (-z) = (a - a) + i(b - b) = 0,$$

showing that complex numbers form a commutative group with respect to addition. For the multiplicative inverse, it is convenient to use complex conjugates.

2.2. **The complex conjugate.** The *complex conjugate* of $z = a + ib$ is given by

$$\bar{z} = a - ib,$$

from where we have formulas for the real and imaginary parts:

$$a = \operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad b = \operatorname{Im} z = \frac{z - \bar{z}}{2i}.$$

We also have the important formula

$$z\bar{z} = (a + ib)(a - ib) = a^2 + b^2. \tag{2.1}$$

The conjugation defines both additive and multiplicative *automorphism* in the sense that

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2,$$

and is an *involution*, i.e. applying conjugation twice returns z:

$$\bar{\bar{z}} = z.$$

These properties of the conjugation often significantly simplify computations.

2.3. **Division.** We can use conjugates and formula (2.1) for division:

$$\frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} = \frac{(a_1 a_2 + b_1 b_2) + i(a_2 b_1 - a_1 b_2)}{a_2^2 + b_2^2}.$$

Thus we can always divide by the complex number $a_2 + ib_2$ different from zero for which $a_2^2 + b_2^2 \neq 0$.

In particular, every $z = a + ib \neq 0$ has the *multiplicative inverse*

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a_2 - ib_2}{a_2^2 + b_2^2} \implies zz^{-1} = z^{-1}z = 1.$$

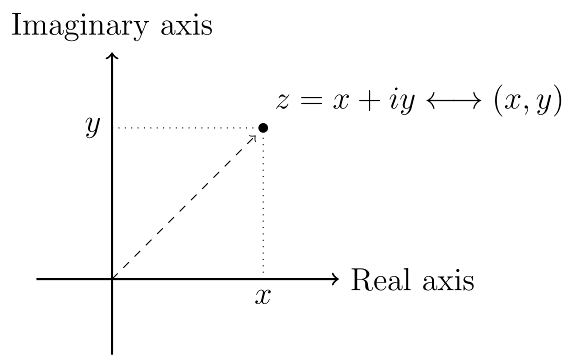
Finally, the distributive law holds as well for complex numbers:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

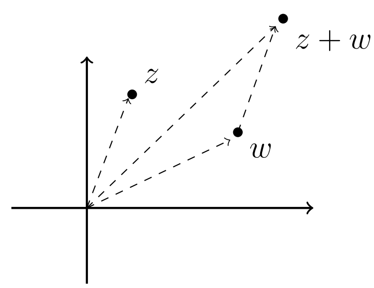
Consequently, complex numbers together with addition and multiplication form a *field*, denoted by $\mathbb{C}$.

3. The complex plane

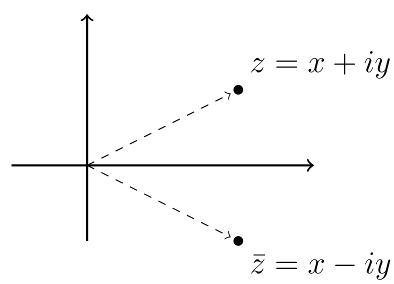
3.1. **Cartesian coordinates.** Since formally the complex number $z = x + iy$ is the pair (x, y), we can identify z with the point (x, y) of the 2-dimensional coordinate plane:



With that geometric interpretation, the addition of complex numbers corresponds to the vector addition:

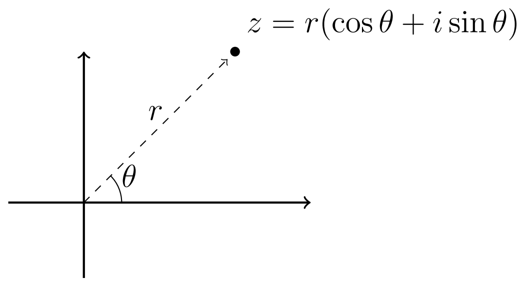


The conjugation $z \to \bar{z}$ of complex numbers corresponds geometrically to the reflection about the x-axis:



While cartesian coordinates are convenient for addition, multiplication is move conveniently expressed by means of the polar form.

3.2. **The polar form.** Using *polar coordinates* in the complex plane, any complex number $z = a + ib$ can be written in its *polar form* as



where

$$r = |z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}} \geq 0 \tag{3.1}$$

is the *modulus* or *absolute value* of z and θ is the *argument* of z, defined for $z \neq 0$ and satsifying

$$\cos\theta = \frac{a}{r}, \quad \sin\theta = \frac{b}{r}, \quad \tan\theta = \frac{b}{a} \text{ (for } a \neq 0).$$

Note that for $z = 0$, the argument θ is not defined, and for $z \neq 0$, the argument is *not unique*! It is defined up to a multiple of 2π. In particular, for $z \neq 0$, we can always choose θ uniquely in the interval $(-\pi, \pi]$, which we call the *principal argument* and denote by $\mathsf{Arg}\, z$. By $\mathsf{arg}\, z$ we denote the set of all possible arguments, i.e.

$$\mathsf{arg}\, z = \{\mathsf{Arg}\, z + 2\pi k : k \in \mathbb{Z}\}.$$

The modulus $|z|$ given by (3.1) satisfy the important properties of a norm:

(1) positivity: $|z| \geq 0$ and $|z| = 0 \iff z = 0$;
(2) multiplicativity: $|zw| = |z||w|$;
(3) triangle inequality: $|z + w| \leq |z| + |w|$.

Proof. Here (1) follows directly from (3.1) and (2),(3) follow from the automorphism properties of the conjugation

$$|zw| = \sqrt{zw\overline{zw}} = \sqrt{zw\bar{z}\bar{w}} = \sqrt{z\bar{z}}\sqrt{w\bar{w}} = |z||w|,$$

$$|z + w|^2 = (z + w)\overline{(z + w)} = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w$$

$$= |z|^2 + |w|^2 + 2\mathrm{Re}(z\bar{w}) \leq |z|^2 + |w|^2 + 2|z\bar{w}| = |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2,$$

where we used the inequality

$$\mathsf{Re}\, \zeta = a \leq \sqrt{a^2 + b^2} = |\zeta|$$

for any complex number $\zeta = a + ib$. □

3.3. **Euler's formula.** We can use the *Euler's formula*

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3.2}$$

to rewrite the polar form as

$$z = re^{i\theta}.$$

Here (3.2) can be viewed as the definition extending the exponential function of imaginary arguments $i\theta$, preserving the basic homomorphism property

$$e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi},$$

which expands to

$$cos(\theta+\varphi) + i\sin(\theta+\varphi) = (cos\theta + i\sin\theta)(cos\varphi + i\sin\varphi), \tag{3.3}$$

with the right-hand side

$$(\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\sin\theta\cos\varphi + \cos\theta\sin\varphi)$$

equal to the left-hand side of (3.3) by standard trigonometric identities.

In polar form, multiplication and division look particularly simple:

$$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}, \quad \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}.$$

The first formula can be iterated to obtain *de Moivre's formula*:

$$(re^{i\theta})^n = r^ne^{in\theta},$$

which can be used for calculating the *nth roots*:

(3.4)

$$\sqrt[n]{re^{i\theta}} = \sqrt[n]{r}e^{i\frac{\theta+2\pi k}{n}} = \sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right), \quad k\in\mathbb{Z}.$$

Note that θ is always defined up to a multiple of 2π, i.e.

$$e^{i\theta} = e^{i\varphi} \iff \theta = \varphi + 2\pi k, \quad k\in\mathbb{Z},$$

and hence the addition of $2\pi k$ is essential in the formula, but it suffices to take in (3.4) only finitely many values

$$k = 0, \dots, n-1,$$

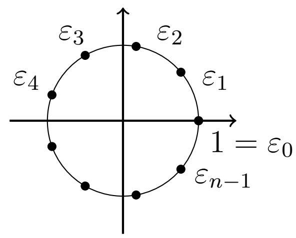
to obtain all n values of the nth root $\sqrt[n]{z}$.

These n values have all the same modulus, while their arguments are distinct for $z \neq 0$. Thus, the nth root of $z \neq 0$ has precisely n values all contained in the circle with center 0 and radius $\sqrt[n]{|z|}$.

An important special case is that of the nth roots of unity, which are the nth roots of 1, i.e. complex numbers z satisfying $z^n = 1$. From (3.4) for $z = 1$, we compute all nth roots of unity as

$$\varepsilon_k := e^{i\frac{2\pi k}{n}}, \quad k = 0, 1, \ldots, n-1. \tag{3.5}$$

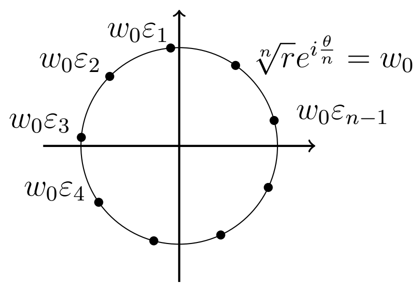
Geometrically the roots of unity are located on the unit circle:



Using (3.5), we can rewrite (3.4) as

$$\sqrt[n]{z} = \sqrt[n]{r} e^{i\frac{\theta}{n}} \varepsilon_k, \quad k = 0, 1, \ldots, n-1,$$

i.e. all n values of the root $\sqrt[n]{z}$ can be obtained by taking one root's value $w_0 := \sqrt[n]{r} e^{i\frac{\theta}{n}}$ and multiplying with all nth roots of unity ε_k:



3.4. **Completing our example of using Cardano's formula.** We can now complete our calculation in the example of using Cardano's formula in §1.3. We have the polar form

$$i = e^{i\frac{\pi}{2}}$$

and hence

$$u = \sqrt[3]{i} = \cos\frac{\pi/2 + 2\pi k}{3} + i \sin\frac{\pi/2 + 2\pi k}{3}, \quad k = 0, 1, 2.$$

Calculating we obtain the values

$$u = \frac{\sqrt{3}}{2} + i\frac{1}{2}, \quad -\frac{\sqrt{3}}{2} + i\frac{1}{2}, \quad -i.$$

Similarly,

$$v = \sqrt[3]{-i} = \cos\frac{-\pi/2 + 2\pi k}{3} + i \sin\frac{-\pi/2 + 2\pi k}{3}, \quad k = 0, 1, 2,$$

which yields

$$v = \frac{\sqrt{3}}{2} - i\frac{1}{2}, \quad -\frac{\sqrt{3}}{2} - i\frac{1}{2}, \quad i.$$

Finally, according to the rule (1.2), we only consider $x = u + v$ for u, v satisfying $uv = -p/3$, i.e. $uv = 1$ in our case. Hence we can only combine the first, second and third values of u with the first, second and third values of v respectively, which yields the three solutions

$$x = u + v = \sqrt{3}, \quad -\sqrt{3}, \quad 0.$$

Note that without the rule (1.2), we would e.g. add the first value of u with the third value of v giving

$$x = \frac{\sqrt{3}}{2} + i\frac{3}{2},$$

which is not a solution of our equation (1.1).

4. Elementary functions of a complex variable

4.1. **Polynomials and rational functions.** Similar to real polynomials, using complex addition and multiplication, we can define a *complex polynomial function*

$$f(z) = a_n z^n + \ldots + a_1 z + a_0, \quad a_0, \ldots, a_n, z \in \mathbb{C},$$

which has degree n if $a_n \neq 0$. In particular, a polynomial of degree 1 is an affine function $f(z) = a_1 z + a_0$.

A *rational function* is a ratio of two polynomials

$$f(z) = \frac{p(z)}{q(z)}, \quad q \not\equiv 0,$$

which is defined whenever q is not zero, i.e. for $z \in \mathbb{C}$ with $q(z) \neq 0$.

4.2. **Exponential and logarithm functions.** The exponential function of complex variable $z = x + iy$ can be defined by Euler's formula

$$e^{x+iy} := e^x(\cos y + i \sin y) = e^x e^{iy}.$$

For $y = 0$, we obtain the real exponential function e^x, i.e. e^z extends e^x to the complex plane.

The fundamental homomorphism property $e^{x_1+x_2} = e^{x_1}e^{x_2}$ extends to complex variables:

$$e^{(x_1+iy_1)+(x_2+iy_2)} = e^{x_1+x_2}e^{i(y_1+y_2)} = e^{x_1}e^{x_2}e^{iy_1}e^{iy_2} = e^{x_1+iy_1}e^{x_2+iy_2},$$

i.e.

$$e^{z_1+z_2} = e^{z_1}e^{z_2}, \quad z_1, z_2 \in \mathbb{C}.$$

The (natural) logarithm function is defined as inverse of the exponential function, i.e.

$$w = \log z \iff z = e^w.$$

Writing $w = x + iy$ and $z = re^{i\theta}$,

$$z = e^w \iff re^{i\theta} = e^{x+iy} = e^x e^{iy} \iff r = e^x, y = \theta + 2\pi k, k \in \mathbb{Z}.$$

Since $r = |z|$ and $\{\theta + 2\pi k, k \in \mathbb{Z}\} = \arg(z)$, we obtain the explicit formula for the logarithm function

$$\log z = \ln|z| + i \arg z, \quad z \in \mathbb{C} \setminus \{0\}.$$

Note that $\log z$ is multi-valued since $\arg z$ is! Taking the principal value $\mathsf{Arg} z$ we can define the *principal value of the logarithm*

$$\mathsf{Log} z := \ln|z| + i\mathsf{Arg} z.$$

4.3. **General power function.** Using exponential and logarithm we can define the *general power*

$$z^w := e^{w \log z}, \quad z, w \in \mathbb{C}, z \neq 0.$$

Since $\log z$ is multi-valued, so is z^w, at least a priori. That is, it may happen that after substituting multiple values of $w \log z$ into the exponential function, the result may be single-valued. This happens when $w = n$ is an integer, in which case all values coincide with the nth power of $z = re^{i\theta}$:

$$z^n = e^{n \log z} = e^{n(\ln r + i\theta z + 2\pi ik)} = e^{\ln r^n} e^{ni\theta} e^{2\pi ikn} = (re^{i\theta})^n = z^n, \quad k \in \mathbb{Z},$$

i.e. the integer power agrees with the usual definition of the power. More generally, a rational power of $z = re^{i\theta}$ attains finitely many values that can be computed as roots:

$$z^{p/q} = e^{p \log z/q} = e^{\frac{p}{q}(\ln r + i\theta z + 2\pi ik)} = e^{\ln r^{p/q}} e^{\frac{p}{q}(i\theta + 2\pi ik)} = (\sqrt[q]{re^{i\theta}})^p = (\sqrt[q]{z})^p.$$

Here some of the pth powers of different values of $\sqrt[q]{z}$ may coincide when p and q have nontrivial common divisors.

Example 4.1. Writing $z = -1$ in the polar form $-1 = e^{i\pi}$ we have

$$(-1)^{2/4} = (\sqrt[4]{-1})^2 = (e^{i(\pi+2\pi ik)/4})^2 = e^{2i(\pi+2\pi ik)/4} = e^{i(\pi+2\pi ik)/2} = \pm e^{i\pi/2} = \pm i,$$

since $e^{i\pi k} = \pm 1$ for $k \in \mathbb{Z}$. To obtain all 4 roots $\sqrt[4]{-1}$ we need to take $k = 0, 1, 2, 3$ but only 2 values remains after taking the square.

4.4. **Trigonometric and inverse trigonometric functions.** In real analysis, trigonometric functions sin and cos are very different from the exponential functions. In complex analysis, they are closely related. From Euler's formula, we have

$$e^{ix} = \cos x + i \sin x, \quad e^{-ix} = \cos x - i \sin x.$$

Adding and subtracting, we obtain

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Using these formulas we can extend sin and cos to arbitrary complex argument z:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{4.1}$$

Now standard trigonometric identities can be proved directly, e.g.

$$\cos^2 z + \sin^2 z = \frac{(e^{iz} + e^{-iz})^2}{4} + \frac{(e^{iz} - e^{-iz})^2}{-4}$$
$$= \frac{(e^{2iz} + 2 + e^{-2iz}) - (e^{2iz} - 2 + e^{-2iz})}{4} = 1.$$

Restricting cosine and sine to the imaginary axis $z = iy$, we obtain respectively the *hyperbolic cosine and sine*:

$$\cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh y, \quad \sin(iy) = \frac{e^{-y} - e^{y}}{2i} = i \sinh y.$$

This shows that the behavior of cos and sin along the imaginary axis is very different from the real axis. In particular, both cos and sin are unbounded on the y-axis because the hyperbolic cosine and sine are unbounded on the real axis.

Using the exponential function e^z for $z \in \mathbb{C}$, we also can define the hyperbolic sine and cosine for complex arguments reusing the same formulas:

$$\cosh z := \frac{e^{z} + e^{-z}}{2}, \quad \sinh z := \frac{e^{z} - e^{-z}}{2}.$$

Finally, formulas (4.1) allow to compute the *inverse trigonometric* functions, e.g.

$$w = \arccos z \iff z = \cos w = \frac{e^{iw} + e^{-iw}}{2} \iff e^{2iw} - 2ze^{iw} + 1 = 0.$$

Solving the quadratic equation, we obtain an explicit formula for arccos z:

$$e^{iw} = z + \sqrt{z^2 - 1} \iff w = \arccos z = \frac{1}{i} \log(z + \sqrt{z^2 - 1}). \tag{4.2}$$

Note that we don't need $\pm$ before the square root when solving the quadratic equation, since the square root of a complex number already takes 2 values of the form $\pm a$, where a is one of the root values. In addition, taking log in (4.2) yields infinitely many values for each value of the square root.

5. Metric and topology in complex plane

5.1. **Metric space structure.** We have seen that the modulus $|z|$ defines a *norm* on $\mathbb{C}$. Then, every norm defines a metric, in our case the distance between $z, w \in \mathbb{C}$ is given by

$$d(z,w) = |z-w|.$$

We can now use the properties of the norm to show that the standard metric axioms are satisfied:

(1) positivity: $|z-w| \geq 0$ and $|z-w| = 0 \iff z = w$;
(2) symmetry: $|z-w| = |w-z|$;
(3) triangle inequality: $|z_1 - z_3| \leq |z_1 - z_2| + |z_2 - z_3|$.

5.2. **Open disks.** In the sequel denote by

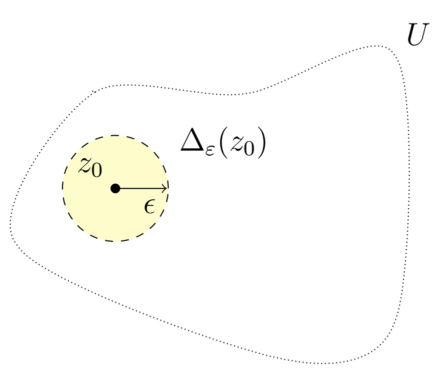
$$\Delta_r(a) := \{z \in \mathbb{C} : |z-a| < r\}$$

the (open) *disk* with center $a \in \mathbb{C}$ and radius $r > 0$.

5.3. **Open sets.**

Definition 5.1. (1) A subset $U \subset \mathbb{C}$ is *open (in $\mathbb{C}$)* if every point $z_0 \in U$ can be surrounded by disk $\Delta_\varepsilon(z_0) \subset U$ for some $\varepsilon > 0$.
(2) More generally, a subset $A \subset S$ is *open in a set $S \subset \mathbb{C}$* (briefly *open in S*, also *open in the relative topology of S*) if for every $z_0 \in A$, there exists $\varepsilon > 0$ with

$$\Delta_\varepsilon(z_0) \cap S \subset A.$$



It is straightforward to check that arbitrary unions $\bigcup_{\alpha\in A} U_\alpha$ and finite intersections $U_1 \cap \cdots \cap U_n$ of open sets (in a set S) are open (in a set S). On the other hand, infinite intersections of open sets may not be open.

Example 5.2. Every open disk $\Delta_r(a)$ with $r > 0$ is open. Indeed, for every $z_0 \in \Delta_r(a)$, take $\varepsilon := r - |z_0 - a|$. Then

$$z \in \Delta_\varepsilon(z_0) \implies |z - a| \le |z - z_0| + |z_0 - a| < \varepsilon + |z_0 - a| = r,$$

proving that z_0 is surrounded by the disk $\Delta_\varepsilon(z_0) \subset \Delta_r(a)$.

Example 5.3. The infinite intersection of open disks

$$\bigcap_{n=1}^{\infty} \Delta_{1/n}(0) = \{0\}$$

is not open, since 0 cannot be surrounded by a disk $\Delta_\varepsilon(0)$ with $\varepsilon > 0$ contained in this intersection.

Example 5.4. Any interval (a, b) is *open* in $\mathbb{R}$ but not in $\mathbb{C}$. Indeed, for every $x \in (a, b)$, taking $\varepsilon := \min(|x - a|, |x - b|)$, we have

$$\mathbb{R} \cap \Delta_\varepsilon(x) \subset (a, b),$$

proving that (a, b) is open in $\mathbb{R}$. On the other hand, (a, b) cannot contain an entire disk $\Delta_\varepsilon(x)$, $\varepsilon > 0$, proving that (a, b) is not open (in $\mathbb{C}$).

5.4. **Closed sets.** A set $F \subset S$ is *closed in* S if $S \setminus F$ is open in S. Using analogous properties of open sets it is easy to see that arbitrary intersections and finite unions of closed sets in S are closed in S. The proof is based on the formulas

$$S \setminus (\bigcap_{\alpha\in A} F_\alpha) = \bigcup_{\alpha\in A} (S \setminus F_\alpha), \quad S \setminus (F_1 \cup \cdots \cup F_n) = (S \setminus F_1) \cap \cdots \cap (S \setminus F_n).$$

5.5. **Interior, exterior and boundary of a subset.** The following fundamental notions are defined for every subset in $\mathbb{C}$:

Definition 5.5. Let $A \subset \mathbb{C}$ be a subset and $z_0 \in \mathbb{C}$.

(1) z is an *interior point* of A if $\Delta_\varepsilon(z) \subset A$ for some $\varepsilon > 0$; the *interior* of A is the set of all interior points;
(2) z is an *exterior point* of A if $\Delta_\varepsilon(z) \subset \mathbb{C} \setminus A$ for some $\varepsilon > 0$; the *exterior* of A is the set of all exterior points;
(3) z is a *boundary point* of A if it neither interior nor exterior, i.e. if for all $\varepsilon > 0$, the disk $\Delta_\varepsilon(z)$ contains points of both A and the complement $\mathbb{C} \setminus A$; the *boundary* ∂A of A is the set of all boundary points;
(4) the *closure* $\overline{A}$ of A is the union of its interior and boundary.

It follows directly that interior and exterior are unions of disks, hence is always open, while the closure is the complement of the exterior, hence is always closed.

Example 5.6. Let $A = \Delta_r(a) = \{z : |z-a| < r\}$ be a disk. Then the interior of A is A, the exterior is $\{z : |z-a| > r\}$, the boundary is the circle

$$\partial A = \{z : |z-a| = r\},$$

and the closure is the closed disk

$$\overline{A} = \{z : |z-a| \le r\}.$$

5.6. **Connected sets.**

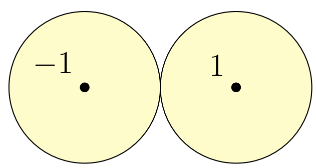
Definition 5.7. A set $S \subset \mathbb{C}$ is *connected* if it cannot be covered by disjoint open sets U and V such that $U \cap S \neq \emptyset$, $V \cap S \neq \emptyset$.

Example 5.8. The union of disks

$$\Delta_1(-1) \cup \Delta_1(1)$$

is *not connected*, because it is covered by the disjoint open sets

$$U = \Delta_1(-1), \quad V = \Delta_1(1) :$$



Lemma 5.9. *Any closed interval $I = [a,b] \subset \mathbb{R}$ is connected.*

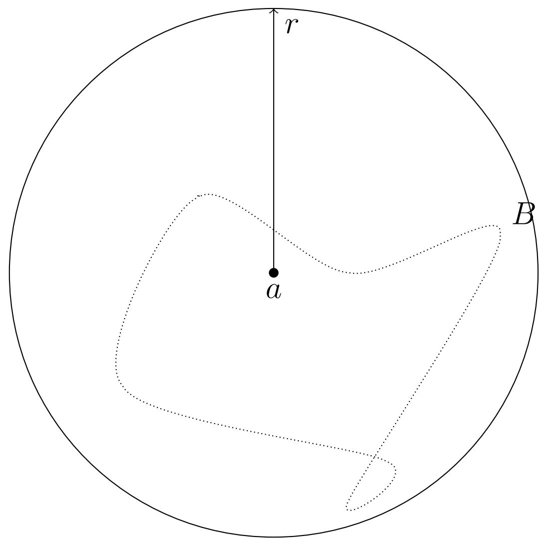
Proof. Assume by contradiction, I can be covered by disjoint *open* subsets U, V as in Definition 5.7. Without loss of generality, $b \in V$ and set

$$c := \sup(U \cap [a,b]) \in [a,b]. \tag{5.1}$$

Then either $c \in U$ or $c \in V$. If $c \in U$, since U is open, there is a disk $\Delta_\varepsilon(c) \subset U$, which contradicts (5.1). Alternatively, if $c \in V$, since V is open, there is a disk $\Delta_\varepsilon(c) \subset V$, which again contradicts (5.1) since $U \cap V = \emptyset$. □

5.7. **Bounded sets.**

Definition 5.10. A subset $B \subset \mathbb{C}$ is *bounded* if it is contained in some disk $\Delta_r(a)$.



Example 5.11. An interval $[a, b] \subset \mathbb{R}$ and a rectangle

$$\{z : \mathbb{C} : x_1 \leq \operatorname{Re} z \leq x_2, y_1 \leq \operatorname{Im} z \leq y_2\}$$

are bounded, while $\mathbb{R}$, $\mathbb{C}$ and the upper half-plane

$$\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

are not bounded.

5.8. **Compact sets.**

Definition 5.12. Let $S \subset \mathbb{C}$ be a subset.

(1) An *open cover* of S is a collection of *open* subsets $U_\alpha \subset \mathbb{C}$ whose union contains S, i.e. $S \subset \bigcup_\alpha U_\alpha$.
(2) A *subcover* of a cover $S \subset \bigcup_{\alpha \in A} U_\alpha$ is any subcollection that still forms a cover, i.e. a subset of indices $B \subset A$ with $S \subset \bigcup_{\alpha \in B} U_\alpha$.
(3) S is said to be *compact* if every *open cover* $S \subset \bigcup_{\alpha \in A} U_\alpha$ has a *finite subcover*, i.e. $S \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ for some n and $\alpha_1, \ldots, \alpha_n \in A$.

Theorem 5.13. *Any* compact *set in* $\mathbb{C}$ *is bounded.*

Proof. Let $S \subset \mathbb{C}$ be compact. Consider the cover $S \subset \bigcup_{z \in S} \Delta_1(z)$ by open disks with radius 1 and centers in S. Since S is *compact*, this cover has a *finite subcover*

$$S \subset \Delta_1(z_1) \cup \cdots \cup \Delta_1(z_n).$$

Without loss of generality, S is nonempty and hence $n \geq 1$. Setting

$$R := \max\{|z_1 - z_k| : 1 \leq k \leq n\}, \quad \text{it follows that}$$

$$z \in \Delta_1(z_k) \implies |z - z_1| \le |z - z_k| + |z_k - z_1| < 2 + R \implies z \in \Delta_{R+2}(z_1)$$

proving $S \subset \Delta_{R+2}(z_1) \implies S$ is *bounded.* □

The role of compactness is passing from local to global as is well illustrated by the following lemma.

Lemma 5.14 (uniform radius for disks)**.** *Let $U \subset \mathbb{C}$ be an* open subset*, $K \subset U$ a* compact *set. Then there exists $r > 0$ such that $\Delta_r(z) \subset U$ for all $z \in K$.*

Proof. By definition, every point z of the open set U is surrounded by a disk $\Delta_{r(z)}(z) \subset U$, where the radius $r(z) > 0$ may depend on z. In particular, the set $K \subset U$ is covered by the disks $\Delta_{r(z)}(z)$ for $z \in K$, and even by the *half-radius* disks

$$K \subset \bigcup_{z \in K} \Delta_{\frac{r(z)}{2}}(z).$$

By definition of compactness, we can pass to a finite subcover

$$K \subset \Delta_{r(z_1)/2}(z_1) \cup \cdots \cup \Delta_{r(z_n)/2}(z_n)$$

for some points $z_1, \ldots, z_n \in K$. Then we claim that

$$r := \min\{r(z_1)/2, \ldots, r(z_n)/2\}$$

satisfies the conclusion of the lemma. Indeed, each $z \in K$ is covered by at least one $\Delta_{r(z_j)/2}(z_j)$ and then, since $r \le r(z_j)/2$,

$$\Delta_r(z) \subset \Delta_{r(z_j)/2}(z) \subset \Delta_{r(z_j)}(z_j) \subset U,$$

where the 2nd inclusion follows from the *triangle inequality*

$$w \in \Delta_{\frac{r(z_j)}{2}}(z) \implies |w - z_j| \le |w - z| + |z - z_j| < \frac{r(z_j)}{2} + \frac{r(z_j)}{2} = r(z_j).$$

□

Without proof we quote the following fundamental result proved in modules on metric spaces and topology:

Theorem 5.15 (Heine-Borel theorem in $\mathbb{C}$)**.** *A subset $S \subset \mathbb{C}$ is* compact *if and only if S is both bounded and closed in $\mathbb{C}$.*

6. Sequences of complex numbers

6.1. **Limits of sequences of complex numbers.** A sequence of complex numbers $(z_n)_{n\geq 1}$ converges to z_0, written

$$z_0 = \lim_{n\to\infty} z_n, \text{ or } z_n \to z_0,\ n \to \infty,$$

if the distance $|z_n - z_0|$ converges to 0. Here the important difference is that $|z_n - z_0|$ is a sequence of *real numbers*, whose convergence is be defined in real analysis. More explicitly, expanding the definition, we obtain:

Definition 6.1. A sequence (z_n) converges to z_0 as $n \to \infty$ whenever for any $\varepsilon > 0$ there exists integer $N \geq 1$ such

$$n \geq N \implies |z_n - z_0| < \varepsilon. \tag{6.1}$$

Example 6.2. Consider the sequence of powers $z_n = a^n$, $n \geq 1$, of a fixed complex number a. If $|a| < 1$, then $z_n \to 0$:

$$|z_n - 0| = |a^n| = |a|^n \to 0, \quad n \to \infty.$$

Convergence of sequences of complex numbers is equivalent to convergence of both sequences of their real and imaginary parts:

Lemma 6.3. *A sequence of complex numbers (z_n) converges to z_0 as $n \to \infty$ if and only if*

$$\mathsf{Re}\, z_n \to \mathsf{Re}\, z_0, \quad \mathsf{Im}\, z_n \to \mathsf{Im}\, z_0, \quad n \to \infty.$$

The proof follows from the estimates

$$|\mathsf{Re}\, z_n - \mathsf{Re}\, z_0| \leq |z_n - z_0|, \quad |\mathsf{Im}\, z_n - \mathsf{Im}\, z_0| \leq |z_n - z_0|,$$

and the formula

$$|z_n - z_0| = \sqrt{|\mathsf{Re}\, z_n - \mathsf{Re}\, z_0|^2 + |\mathsf{Im}\, z_n - \mathsf{Im}\, z_0|^2}.$$

6.2. **Limits of sums, products and ratios of sequences.**

Lemma 6.4. *Let*

$$a_n \to a, \quad b_n \to b, \quad n \to \infty,$$

be convergent sequences. Then

$$(a_n + b_n) \to (a + b), \quad (a_n - b_n) \to (a - b), \quad a_n b_n \to ab, \quad n \to \infty.$$

If in addition $b \neq 0$ and $b_n \neq 0$ for all n, then

$$\frac{a_n}{b_n} \to \frac{a}{b}, \quad n \to \infty.$$

The proof is analogous to the case of real sequences from real analysis.

6.3. **Bounded sequences.**

Definition 6.5. A sequence (z_n) is *bounded* if there exists $C > 0$ with $|z_n| \le C$ for all n.

Lemma 6.6. *Any convergent sequence is bounded.*

Proof. Let $z_n \to z_0$, $n \to \infty$, be a convergent sequence. Taking $\varepsilon = 1$ in Definition 6.1, there exists N such that (6.1) holds. Then the triangle inequality yields

$$|z_n| = |z_n - z_0 + z_0| \le |z_n - z_0| + |z_0| < \varepsilon + |z_0| = 1 + |z_0|$$

for $n \ge N$, and we can take

$$C = 1 + |z_0| + \max\{|z_n| : 1 \le n < N\}$$

in Definition 6.5 to conclude (z_n) is bounded. □

Example 6.7. The sequence of powers $z_n = a^n$ is *unbounded* for $|a| > 1$, since in this case $|a^n| = |a|^n \to \infty$, $n \to \infty$. Therefore, the sequence (z_n) is *divergent*, i.e. cannot have a finite limit.

On the other hand, we can define the notion of convergence to ∞ which is satisfied in this case:

Definition 6.8. A sequence of complex numbers (z_n) is said to converge to ∞ as $n \to \infty$, written

$$z_n \to \infty, \quad n \to \infty,$$

whenever $|z_n| \to \infty$, $n \to \infty$.

6.4. **Cauchy sequences and Cauchy's criterion.** A sequence is convergent if it has a limit. However, the definition of convergence requires knowledge of the limit.

Cauchy's criterion is an elegant way of expressing convergence without referring to the limit.

Definition 6.9. A sequence (z_n) is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists N such that

$$m, n \ge N \implies |z_m - z_n| < \varepsilon.$$

Theorem 6.10 (Cauchy's criterion)**.** *A sequence (z_n) is convergent (converges to some limit) if only if it is a Cauchy sequence.*

This criterion for a complex sequence (z_n) can be derived from the analogous criterion from real analysis for the sequences of real numbers $(\mathsf{Re}\, z_n)$ and $(\mathsf{Im}\, z_n)$.

7. Series of complex numbers

7.1. **Convergence of series.** A complex series is a *formal sum*

$$\sum_{n\geq 1} a_n = a_1 + a_2 + \ldots \tag{7.1}$$

where a_n are arbitrary complex numbers.

Definition 7.1. The sequence of *partial sums* of (7.1) is

$$s_n := a_1 + \ldots + a_n,$$

and the series (7.1) is said to converge to a complex number z_0 whenever the sequence (s_n) of partial sums converges to z_0 as $n \to \infty$. A series which is not convergent to any finite complex number is called *divergent*.

Example 7.2. Let $a \in \mathbb{C}$. The complex *geometric series* $\sum_{n=0}^{\infty} a^n$ converges to $\frac{1}{1-a}$ when $|a| < 1$. The proof is based on the formula

$$(1-a)(1+a+\ldots+a^n) = 1 - a^{n+1}.$$

When $|a| < 1$, the right-hand side converges to 1, which implies $s_n \to \frac{1}{1-a}$, $n \to \infty$, for the sequence

$$s_n = 1 + a + \ldots + a^n$$

of partial sums, as desired.

7.2. ***n*th term test.** The nth term test can be convenient when proving divergence.

Theorem 7.3 (nth term test)**.** *If the series $\sum a_n$ is convergent, the nth term a_n must converge to* 0*.*

For a proof, write the nth term as the difference of partial sums

$$a_n = s_n - s_{n-1}$$

and use the convergence of s_n and s_{n-1} to the same limit as $n \to \infty$.

Example 7.4. The geometric series $\sum_{n\geq 0} a^n$ is divergent when $|a| \geq 1$. Indeed, otherwise the nth term test would imply $|a^n| = |a|^n \to 0$, $n \to \infty$, which is not the case since $|a|^n \geq 1$.

7.3. **Cauchy's criterion for series.** The following is a direct consequence of Cauchy's criterion for sequences applied to the sequence of partial sums:

Theorem 7.5. *A series $\sum_n a_n$ is convergent if and only if for every $\varepsilon > 0$ there exists N such that*

$$N \leq m < n \implies |a_{m+1} + \ldots + a_n| < \varepsilon.$$

7.4. **Comparison test.** Comparison test is an important method of reducing the question of convergence of complex series to that of their real-valued majorant series:

Theorem 7.6. *Let* $\sum_n z_n$ *and* $\sum_n x_n$ *be respectively complex and real series satisfying the majorant condition*

$$|z_n| \leq x_n.$$

Then if $\sum x_n$ *is convergent, so is* $\sum z_n$.

Proof. The proof is based on the estimate

$$|z_{m+1} + \ldots + z_n| \leq |z_{m+1}| + \ldots + |z_n| \leq x_{m+1} + \ldots + x_n,$$

and Theorem 7.5. □

7.5. **Absolute convergence.**

Definition 7.7. A series $\sum z_n$ *converges absolutely* if the series of absolute values of its terms $\sum |z_n|$ converges.

Lemma 7.8. *An absolutely convergent series is convergent.*

This follows immediately from the comparison test with the majorant series $\sum |z_n|$.

7.6. **Ratio test.**

Theorem 7.9. *Consider a series* $\sum_n z_n$ *such that for* n *sufficiently large* $z_n \neq 0$ *and there exists a real number* q *satisfying*

$$\left|\frac{z_{n+1}}{z_n}\right| \leq q < 1. \tag{7.2}$$

Then the series $\sum_n z_n$ *converges absolutely. If* $\left|\frac{z_{n+1}}{z_n}\right| \geq 1$ *it diverges.*

Proof. If $\left|\frac{z_{n+1}}{z_n}\right| \geq 1$, then for all $n \geq N$ $|z_n| \geq |z_N| > 0$, violating the nth term test, hence the series diverges. The convergence part is based on comparison with geometric series. If (7.2) holds for all $n \geq N$, then

$$|z_{N+p}| \leq |z_N| q^p, \quad p \geq 1. \tag{7.3}$$

Since $q < 1$, the geometric series $\sum_p |z_N| q^p$ converges and the conclusion follows from the comparison test. □

7.7. **Root test.**

Theorem 7.10. *Consider a series $\sum_n z_n$ such that for every n sufficiently large and some real number q,*

$$\sqrt[n]{|z_n|} \le q < 1, \tag{7.4}$$

then the series converges absolutely. If $\sqrt[n]{|z_n|} \ge 1$ for all n sufficiently large, it diverges.

Proof. (7.4) is equivalent to $|z_n| \le q^n$ and the convergence of $\sum_n q^n$ implies the absolute convergence of $\sum_n z_n$ by comparison test. On the other hand, the estimate $\sqrt[n]{|z_n|} \ge 1$ contradicts the nth term test $z_n \to 0$, $n \to \infty$, hence the series diverges. $\square$

8. Limits and continuity of functions of a complex variable

8.1. **Limit points of subsets.**

Limits of functions can be considered for functions defined on subsets $S \subset \mathbb{C}$ at their *limit points*.

Definition 8.1. A point $a \in \mathbb{C}$ is a limit point of S if every punctured disk $\Delta_\varepsilon(a) \setminus \{a\}$ contains a point of S (which implies that every punctured disk contains infinitely many points of S). Note that a limit point of S does not need to belong to S.

Example 8.2. 0 is the only limit point of the set $\{\frac{1}{n} : n \in \mathbb{Z}\}$.

8.2. **Limits of functions.**

Definition 8.3. If a is a *limit point* of S, then L is the limit of a function $f \colon S \to \mathbb{C}$, written as

$$L = \lim_{z \to a} f(z)$$

if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - a| < \delta, z \neq a, z \in S \implies |f(z) - L| < \varepsilon.$$

If a is a limit point of S, the set on the left-hand side

$$\{z : |z - a| < \delta, z \neq a, z \in S\} = (\Delta_\delta(a) \setminus \{a\}) \cap S$$

is always nonempty.

The following is an important example illustrating the depence on the domain of the function:

Example 8.4. Recall that $\mathsf{Arg} z$ is the principal argument of $z \neq 0$ defined by the condition $-\pi < \mathsf{Arg} z \le \pi$. Consider the upper and lower half-planes

$$S_+ := \{z \in \mathbb{C} : \mathsf{Im}\, z > 0\}, \quad S_- := \{z \in \mathbb{C} : \mathsf{Im}\, z < 0\},$$

and functions

$$f: S_+ \to \mathbb{C}, \quad f(z) = \mathsf{Arg} z,$$

and

$$g: S_- \to \mathbb{C}, \quad g(z) = \mathsf{Arg} z.$$

Then

$$\lim_{z\to 1} f(z) = \lim_{z\to 1} g(z) = 0,$$

while

$$\lim_{z\to -1} f(z) = \pi, \quad \lim_{z\to -1} g(z) = -\pi.$$

Also note that the value $\mathsf{Arg}(-1) = \pi$ does not matter for the limits of f and g since $-1 \notin S_-$, $-1 \notin S_+$.

Similar to sequences, we have analogous algebraic properties of limits of functions:

Lemma 8.5. *Let $f, g: S \to \mathbb{C}$ be functions defined over subset $S \subset \mathbb{C}$ with limit point a. Suppose there exist limits*

$$L = \lim_{z\to a} f(z), \quad M = \lim_{z\to a} g(z).$$

Then

$$(L+M) = \lim_{z\to a}(f(z)+g(z)), \quad (L-M) = \lim_{z\to a}(f(z)-g(z)), \quad LM = \lim_{z\to a} f(z)g(z).$$

If in addition $M \neq 0$ and $g(z) \neq 0$ for all $z \in S$, then

$$\frac{L}{M} = \lim_{z\to a} \frac{f(z)}{g(z)}.$$

8.3. **Continuous functions.**

Definition 8.6. A function $f: S \to \mathbb{C}$ is continuous at $z_0 \in S$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$z \in S, |z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon,$$

or equivalently,

$$f(S \cap \Delta_\delta(z_0)) \subset \Delta_\varepsilon(f(z_0)).$$

The function f is continuous on S whenever it is continuous at every $z_0 \in S$.

As a direct consequence of the defintions, we obtain:

Lemma 8.7. *A function $f: S \to \mathbb{C}$ is continuous on S if and only if $\lim_{z\to z_0} f(z) = f(z_0)$ for every limit point z_0 of S.*

Lemma 8.8. *Let $f, g\colon S \to \mathbb{C}$ be functions defined over subset $S \subset \mathbb{C}$ that are continuous at some $z_0 \in S$. Then $f+g$, $f-g$, fg are also continuous at z_0. If in addition $g(z) \neq 0$ for all $z \in S$, then the ratio f/g is also continuous at z_0.*

Lemma 8.9 (Composition of continuous functions)**.** *Let $S, T \subset \mathbb{C}$ be subsets and $f\colon S \to \mathbb{C}$, $g\colon T \to \mathbb{C}$ be functions such that $f(S) \subset T$. Suppose that f is continuous at $z_0 \in S$ and g is continuous at $w_0 = f(z_0)$. Then the composition function*

$$g \circ f\colon S \to \mathbb{C}, \quad (g \circ f)(z) := g(f(z))$$

is continuous at z_0.

Proof. Let $\varepsilon > 0$ be given. Since g is continuous at $w_0 = f(z_0)$, there exists $\eta > 0$ such that

$$w \in T, |w - w_0| < \eta \implies |g(w) - g(w_0)| < \varepsilon.$$

Since f is continuous at z_0, there exists $\delta > 0$ such that

$$z \in S, |z - z_0| < \delta \implies |f(z) - f(z_0)| < \eta.$$

Combining above implications, we obtain

$$z \in S, |z - z_0| < \delta \implies |g(f(z)) - g(f(z_0))| < \varepsilon$$

as desired. □

8.4. **Examples of continuous functions.**

Example 8.10. The constant function $f(z) = c$ and the identity function $f(z) = z$ are continuous by a direct verification. Taking their sums and products, it follows from Lemma 8.8 that a polynomial function $p(z) = a_n z^n + \ldots + a_0$ is continuous on $\mathbb{C}$. Taking ratios, it follows that a rational function $\frac{p(z)}{q(z)}$ is continuous on its domain $\{z : q(z) \neq 0\}$.

Example 8.11. The exponential function $f(z) = e^z$ is continuous on $\mathbb{C}$. Indeed, the real and imaginary parts $\mathsf{Re}\, z, \mathsf{Im}\, z$ are continuous functions of z since

$$|z - z_0| < \varepsilon \implies |\mathsf{Re}\, z - \mathsf{Re}\, z_0|, |\mathsf{Im}\, z - \mathsf{Im}\, z_0| < \varepsilon.$$

From real analysis, we know that $e^x, \sin x, \cos x$ are continuous, hence also their compositions with $\mathsf{Re}\, z$ and $\mathsf{Im}\, z$,

$$e^{\mathsf{Re}\, z}, \quad \cos(\mathsf{Im}\, z), \quad \sin(\mathsf{Im}\, z)$$

are continuous, and hence Lemma 8.8 implies that

$$e^z = e^{\mathsf{Re}\, z}(\cos(\mathsf{Im}\, z) + i \sin(\mathsf{Im}\, z))$$

is continuous.

Example 8.12. The modulus function $f(z) = |z|$ is continuous. Indeed, using triangle inequalities

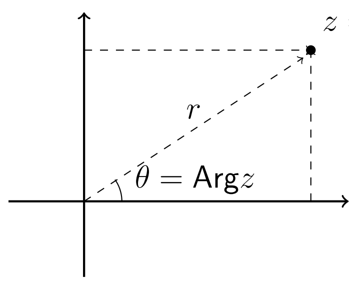
$$|z| \le |z_0| + |z - z_0| \implies |z| - |z_0| \le |z - z_0|,$$
$$|z_0| \le |z| + |z_0 - z| \implies |z_0| - |z| \le |z - z_0|$$

and hence

$$|f(z) - f(z_0)| = ||z| - |z_0|| \le |z - z_0|.$$

Then $|z - z_0 < \varepsilon \implies |f(z) - f(z_0)| < \varepsilon$ proves the continuity of f.

Example 8.13. The principal argument $-\pi < \mathsf{Arg} z \le \pi$ of $z \ne 0$ is continuous at every $z \in \mathbb{C} \setminus (-\infty, 0]$ and is discontinuous (i.e. not continuous) at every $z \in (-\infty, 0)$.



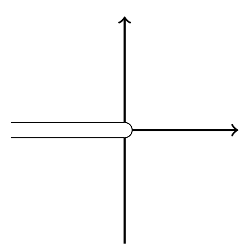
Indeed, $\mathsf{Arg} z$ away from $(-\infty, 0]$ can be computed as

$$\mathsf{Arg} z = \begin{cases} \cos^{-1} \frac{\mathsf{Re}\, z}{|z|} & \text{for } \mathsf{Im}\, z > 0 \\ \sin^{-1} \frac{\mathsf{Im}\, z}{|z|} & \text{for } \mathsf{Re}\, z > 0 \\ -\cos^{-1} \frac{\mathsf{Re}\, z}{|z|} & \text{for } \mathsf{Im}\, z < 0 \end{cases},$$

where each function is continuous as composition of continuous functions.

On the other hand, $\mathsf{Arg} z = \pi$ for $z \in (-\infty, 0)$, while the restriction $\mathsf{Arg} z|_{\mathsf{Im}\, z < 0}$ has limit $-\pi$ at each $z \in (-\infty, 0)$, implying that $\mathsf{Arg} z$ is not continuous at $z \in (-\infty, 0)$.

Thus, to obtain a continuous function, we need to restrict $\mathsf{Arg} z$ to $\mathbb{C} \setminus (-\infty, 0]$, i.e. on the complex plane with the negative real axis cut out:



8.5. **Branches of multi-valued function.** In the previous example we constructed a so-called *branch* of the multi-valued argument $\arg z$:

Definition 8.14. A *branch* of a complex multi-valued function $F(z)$ in on a subset S is any *continuous* function $f\colon S \to \mathbb{C}$ with $f(z) \in F(z)$ for all $z \in S$.

Example 8.15. Recall that the principal logarithm is given by

$$\mathsf{Log} z = \ln|z| + i\mathsf{Arg} z.$$

Since $\mathsf{Arg} z$ is continuous on $S := \mathbb{C} \setminus (-\infty, 0]$, $\mathsf{Log} z$ is also continuous on S and hence is a branch of $\log z$.

Furthermore, since $\arg z$ is defined up to a multiple of 2π,

$$f_k(z) = \mathsf{Log} z + 2\pi i k \in \ln|z| + i \arg z = \log z$$

is a branch of $\log z$ for every $k \in \mathbb{Z}$.

Example 8.16. With $S := \mathbb{C} \setminus (-\infty, 0]$ as before, we can define branches on S of the nth root $\sqrt[n]{z}$ by

$$f_{n,k}(z) := \sqrt[n]{|z|}e^{\frac{i}{n}(\mathsf{Arg} z + 2\pi k)}, \quad z \in S,$$

for each fixed n and $k = 0, 1, \ldots, n-1$. Here $f_{n,k}$ is the kth branch of the multi-valued nth root.

8.6. **Open set criterion for continuity.** Recall that for every function $f\colon X \to Y$ and subset $U \subset Y$, its *preimage* is defined by

$$f^{-1}(U) = \{x \in X : f(x) \in U\} \subset X.$$

Theorem 8.17 (open set criterion for continuity)**.** *A function $f\colon S \to \mathbb{C}$ with $S \subset \mathbb{C}$ is* continuous *if and only if for every* open *subset $U \subset \mathbb{C}$, its preimage $f^{-1}(U)$ is (relatively)* open *in S.*

Note the important distinction between being open (in $\mathbb{C}$) and open in S. In general, the preimage $f^{-1}(U)$ may not be open in $\mathbb{C}$. For example take $S := \mathbb{R}$, $f(z) = z$, $U = \mathbb{C}$, then $f^{-1}(U) = \mathbb{R}$ is open in S but not in $\mathbb{C}$.

Proof. $\Longrightarrow$: Assume f is continuous, let $U \subset \mathbb{C}$ be open and fix $z_0 \in f^{-1}(U)$. By definition of preimage, $f(z_0) \in U$ and since U is open, there exists a disk $\Delta_\varepsilon(f(z_0)) \subset U$. But then the continuity of f implies

$$\exists \delta > 0 \quad f(S \cap \Delta_\delta(z_0)) \subset \Delta_\varepsilon(f(z_0)) \subset U,$$

and hence $S \cap \Delta_\delta(z_0) \subset f^{-1}(U)$ proving $f^{-1}(U)$ is open in S since $z_0 \in f^{-1}(U)$ is arbitrary.

$\Longleftarrow$: Suppose "U is open" implies "$f^{-1}(U)$ is open in S" for all U, and fix $z_0 \in S$, $\varepsilon > 0$. Since every disk $U := \Delta_\varepsilon(f(z_0))$ is open, its preimage

$$f^{-1}(U) = f^{-1}(\Delta_\varepsilon(f(z_0))) \ni z_0$$

is open in S by our assumption, hence there exists a disk $\Delta_\delta(z_0)$ with $\Delta_\delta(z_0) \cap S \subset f^{-1}(U)$ which implies

$$f(S \cap \Delta_\delta(z_0)) \subset \Delta_\varepsilon(f(z_0)),$$

hence f is *continuous* at z_0 since $\varepsilon > 0$ was arbitrary. Since $z_0 \in S$ is arbitrary, f is continuous. □

8.7. **Continuity and connectedness.**

Theorem 8.18 (continuous images of connected sets are connected)**.** *If* $f\colon S \to \mathbb{C}$ *is a* continuous *function and* S *is* connected*, then the* image $f(S)$ *is also* connected*.*

Proof. Suppose by contradiction, $f(X)$ is contained in a union of disjoint *open* sets A, B with

$$A \cap f(X) \neq \emptyset, \quad B \cap f(X) \neq \emptyset. \tag{8.1}$$

Then the *continuity* of f implies that the sets

$$U := f^{-1}(A), \quad V := f^{-1}(B),$$

are *open* in S. In addition, (8.1) implies $U, V \neq \varnothing$, and $A \cap B = \emptyset$ implies $U \cap V = \emptyset$. Hence S is covered by the disjoint *nonempty open* subsets U, V, which contradicts its *connectedness*. Hence $f(X)$ is *connected*. □

We now provide an application of Theorem 8.18 for which we introduce a stronger notion of path-connectedness that is often easier to use to establish connectedness.

Definition 8.19. Let $S \subset \mathbb{C}$ be a subset.

(1) A *path* in S is any continuous map $\gamma\colon [a_1, a_2] \to S$, where $[a_1, a_2] \subset \mathbb{R}$.
(2) S is called *path-connected* if for any two points $s_1, s_2 \in S$ there exists a path $\gamma\colon [a_1, a_2] \to S$ with $\gamma(a_1) = s_1$, $\gamma(a_2) = s_2$.

Theorem 8.20. *Any path-connected subset of* $\mathbb{C}$ *is connected.*

Proof. Let $S \subset \mathbb{C}$ be path-connected and assume by contradiction it is not connected. Then S is contained in a union of disjoint *open* sets A_1, A_2 with

$$A_1 \cap S \neq \emptyset, \quad A_2 \cap S \neq \emptyset.$$

Fix some $s_1 \in A_1 \cap S$, $s_2 \in A_2 \cap S$. Since S is path-connected, there exists a path $\gamma\colon [a_1, a_2] \to S$ with $\gamma(a_1) = s_1$, $\gamma(a_2) = s_2$. But then the image $\gamma([a_1, a_2])$ is contained in the union of disjoint open sets sets A_1, A_2 with

$$A_1 \cap \gamma([a_1, a_2]) \neq \emptyset, \quad A_2 \cap \gamma([a_1, a_2]) \neq \emptyset,$$

which implies that $\gamma([a_1, a_2])$ is not connected. On the other hand, $\gamma([a_1, a_2])$ is the image of the connected interval $[a_1, a_2]$ under the continuous map γ, hence is connected by Theorem 8.18. Thus we reached a contradiction proving that S is connected.

□

8.8. **Applications of connectedness to branches of multi-valued functions.** We here illustrate how connectedness can be used to compute all branches of $\arg z$. The main argument is given by the following uniqueness lemma:

Lemma 8.21. *Let $S \subset \mathbb{C} \setminus \{0\}$ be connected and $g_1, g_2\colon S \to \mathbb{C}$ branches of* $\arg z$. *Then*

$$g_2 = g_1 + \text{const}.$$

Proof. Let $g := g_2 - g_1$. Then for every $z \in S$,

$$g_1(z), g_2(z) \in \arg z$$

implies

$$g(z) = g_2(z) - g_1(z) = 2\pi k, \quad k \in \mathbb{Z}, \tag{8.2}$$

where a priori k may depend on z. We want to use the connectedness of S to prove that k is in fact independent of $z \in S$. Fix $z_0 \in S$ and $k_0 \in \mathbb{Z}$ with $g(z_0) = 2\pi k_0$ and consider the set

$$U := \{z \in S : g(z) = 2\pi k_0\}.$$

Since g_1, g_2 are continuous on S by the definition of a branch, so is $g(z)$. Then the continuity of g implies that, for any $z_1 \in S$ and $\varepsilon = 2\pi$, there exists $\delta > 0$ such that

$$z \in S, |z - z_1| < \delta \implies |g(z) - g(z_1)| < 2\pi \implies g(z) = g(z_1), \tag{8.3}$$

where we used (8.2), since $|2\pi k| < 2\pi$ for $k \in \mathbb{Z}$ implies $k = 0$. In particular, for $z_1 \in U$, it follows that $g = 2\pi k_0$ on $S \cap \Delta_\delta(z_1)$, i.e.

$$S \cap \Delta_\delta(z_1) \subset U,$$

which proves that U is open in S. Similarly, for $z_1 \notin U$, it follows that $g \neq 2\pi k_0$ on $S \cap \Delta_\delta(z_1)$, i.e.

$$S \cap \Delta_\delta(z_1) \subset S \setminus U,$$

which proves that $S \setminus U$ is also open in S. Hence we have represent $S = U \cup (S \setminus U)$ as disjoint union of open subsets. By continuity of S, these sets cannot be both nonempty. Since $U \neq \emptyset$, it follows that $S \setminus U = \emptyset$, hence $U = S$ proving that $g(z) = 2\pi k_0$ for all $z \in S$ as desired. □

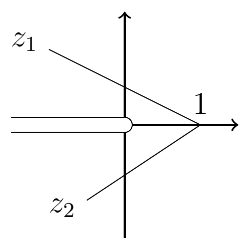
Corollary 8.22. *All branches of* $\arg z$ *on* $S = \mathbb{C} \setminus (-\infty, 0]$ *are of the form*

$$f_k(z) = \mathsf{Arg}(z) + 2\pi k, \quad k \in \mathbb{Z}. \tag{8.4}$$

Proof. It is easy to see that S is path-connected. In fact, for every point $z_1 \in S$, the path

$$\gamma(t) := (1-t)z_1 + 1, \quad t \in [0,1], \implies \gamma(0) = z_1, \gamma(1) = 1$$

and γ is a path in S:



Then combining line segments for any two points z_1, z_2, it follows that S is path-connected. Hence S is connected by Theorem 8.20.

Now, let $f\colon S \to \mathbb{C}$ be any branch of $\mathsf{Arg}z$ and fix $z_0 \in S$. Then $f(z_0) \in arg z_0$ implying

$$f(z_0) = \mathsf{Arg}z_0 + 2\pi k_0 = f_{k_0}(z_0) \tag{8.5}$$

for some $k_0 \in \mathbb{Z}$. Then $f(z) - f_{k_0}(z) = \text{const}$ by Lemma 8.21 and (8.5) implies $f(z) - f_{k_0}(z) = 0$, i.e. the branch f is of the form (8.4) as claimed. □

8.9. **Continuity and compactness.**

Theorem 8.23 (continuous images of compact spaces are compact)**.** *If* $f\colon S \to \mathbb{C}$ *is a* continuous *function and* S *is* compact, *then the image* $f(S)$ *is also* compact.

Proof. To show that $f(S) \subset \mathbb{C}$ is *compact*, consider an *open cover* $f(S) \subset \bigcup_\alpha U_\alpha$. Then each *preimage*

$$f^{-1}(U_\alpha) = \{z : f(z) \in U_\alpha)\} \subset \mathbb{C}$$

is open by the open set criterion of continuity and

$$f(S) \subset \bigcup_\alpha U_\alpha \implies S \subset \bigcup_\alpha f^{-1}(U_\alpha),$$

i.e. the preimages $f^{-1}(U_\alpha)$ form an open cover of S. We now can use the *compactness* of S to conclude the existence of a *finite subcover*

$$S \subset f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_n}) \implies f(S) \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_n},$$

proving the existence of a *finite subcover* for $f(S)$ as claimed. □

8.10. **Uniform continuity.** Uniform continuity is a stronger condition that continuity that is sometimes needed in applications:

Definition 8.24. A function $f\colon S \to \mathbb{C}$ is called *uniformly continuous* if for every $\varepsilon >$ there exists $\delta > 0$ such that

$$z, w \in S,\ |z - w| < \delta \implies |f(z) - f(w)| < \varepsilon. \tag{8.6}$$

It is easy to see from definitions that any *uniformly continuous* function is *continuous* (i.e. continuous at every point). The key difference is that continuity of f is defined at a fixed point z, hence $\delta > 0$ in (8.6) may depend on z. However, on compact sets both notions turn out to be equivalent:

Theorem 8.25 (Uniform continuity of continuous functions on compacta)**.** *Let $S \subset \mathbb{C}$ be compact and $f\colon S \to \mathbb{C}$ be continuous. Then f is* uniformly continuous*.*

Proof. By definition of continuity, for every $z \in S$ and $\varepsilon > 0$, there exists $\delta = \delta(z) > 0$ such that

$$w \in S,\ |z - w| < \delta(z) \implies |f(z) - f(w)| < \frac{\varepsilon}{2}. \tag{8.7}$$

Then

$$S \subset \bigcup_{z \in S} \Delta_{\frac{\delta(z)}{2}}(z)$$

is an open cover, which, by compactness, has a finite subcover

$$S \subset \Delta_{\frac{\delta(z_1)}{2}}(z_1) \cup \ldots \cup \Delta_{\frac{\delta(z_n)}{2}}(z_n).$$

Hence every $z \in S$ is covered by some $\Delta_{\frac{\delta(z_j)}{2}}(z_j)$. Taking $\delta := \min_{1 \le j \le n} \frac{\delta(z_1)}{2}$ we conclude for $w \in S$:

$$|z - w| < \delta \le \frac{\delta(z_j)}{2}(z_j) \implies |w - z_j| \le |w - z| + |z - z_j| < \delta(z_j).$$

Now using (8.7) for z_j yields

$$|f(z_j) - f(z)| < \frac{\varepsilon}{2}, \quad |f(z_j) - f(w)| < \frac{\varepsilon}{2} \implies |f(z) - f(w)| < \varepsilon$$

by the triangle inequality, as desired. □

9. Function sequences and series

9.1. **Terminology.** A function sequence is simply a sequences of functions

$$f_n\colon S \to \mathbb{C},$$

where the domain of definition S is the same for all functions. Similarly, a function series is a series

$$\sum_{n=1}^{\infty} f_n(z)$$

whose terms are functions defined on the same domain.

If the terms of a series are initially not defined on the same domain, the intersection of their domains is to be considered as the maximum subset where all terms are defined:

Example 9.1. The series

$$\sum_{n\geq 1} \frac{1}{z-n}$$

is defined for $z \in S := \mathbb{C} \setminus \mathbb{N}$, which is the intersection of all domains of its terms.

9.2. **Pointwise convergence of function sequences.** There are two main notions of convergence for function sequences and series - pointwise and uniform.

Definition 9.2. A function sequence $f_n\colon S \to \mathbb{C}$ converges *pointwise* to a function $f\colon S \to \mathbb{C}$ (with the same domain) as $n \to \infty$ if for each $z \in S$, their values $f_n(z)$ converge to $f(z)$:

$$\forall z \in S \quad f_n(z) \to f(z), \quad n \to \infty.$$

Example 9.3. The function sequence

$$f_n(z) = z^n, \quad z \in S := \Delta_1(0) \cup \{1\},$$

converges pointwise to the function

$$f(z) = \begin{cases} 0 & z \in \Delta_1(0) \\ 1 & z = 1. \end{cases}$$

On the other hand, since $(-1)^n$ does not have a limit, the same sequence is not pointwise convergent on $S' = \Delta_1(0) \cup \{-1\}$ (where $z = -1$ is the only point where the sequence diverges).

In the above example all functions f_n are continuous on S, whereas their pointwise limit $f(z)$ is *not continuous* at $z = 1$. This phenomenon motivates the stronger notion of *uniform convergence*, which preserves continuity.

9.3. **Uniform convergence of function sequences.**

Definition 9.4. A function sequence $f_n \colon S \to \mathbb{C}$ converges *uniformly* to a function $f \colon S \to \mathbb{C}$ (with the same domain) as $n \to \infty$ if for every $\varepsilon > 0$ there exists N such that

$$\forall z \in S, n \geq N \implies |f_n(z) - f(z)| < \varepsilon. \tag{9.1}$$

The key difference between uniform and pointwise convergence is that the number N in (9.1) is chosen *independently of* z, whereas for a pointwise convergence the choice of N is made for a fixed z, hence may depend on z.

Example 9.5. We have seen that the sequence of powers $f_n(z) := z^n$ converges pointwise to 0 on the unit disk $S = \Delta_1(0)$. In this case (9.1) becomes

$$\forall z\ |z| < 1, n \geq N \implies |z^n| < \varepsilon.$$

In particular, for $n = N$, we would have

$$\forall z\ |z| < 1 \implies |z| < \varepsilon^{1/N},$$

which cannot hold if $\varepsilon < 1 \implies \varepsilon^{1/N} < 1$. This shows that there is no N such that (9.1) holds, hence the convergence is not uniform.

An equivalent way of defining uniform convergence is given by the following lemma following directly from the definition:

Lemma 9.6. *A function sequence $f_n \colon S \to \mathbb{C}$ converges* uniformly *to a function $f \colon S \to \mathbb{C}$ as $n \to \infty$ if and only if*

$$\sup_{z \in S} |f_n(z) - f(z)| \to 0, \quad n \to \infty.$$

Example 9.7. In the above example with $f_n(z) = z^n$, $f(z) = 0$, $z \in S = \Delta_1(0)$, we compute

$$\sup_{z \in S} |f_n(z) - f(z)| = \sup_{|z|<1} |z^n| = 1 \neq 0, \quad n \to \infty.$$

9.4. **Convergence of function series.** Similarly to number series, both pointwise and uniform convergence of function series are reduced to respective convergence for the sequence of their partial sums:

Definition 9.8. A function series $\sum_{n=1}^{\infty} g_n(z)$ converges pointwise (resp. uniformly) to a function $s(z)$ (with the same domain) if the sequence of corresponding partial sums

$$s_n(z) := \sum_{n=1}^{n} g_k(z)$$

converges pointwise (resp. uniformly) to $s(z)$.

9.5. **Weierstrass M-test.** A central result providing uniform convergence of function series is theWeierstrass M-test ("M" is for "majorant"):

Theorem 9.9. *Let $\sum_n f_n(z)$, $z \in S$, and $\sum_n x_n$ be respectively complex function series and real number series satisfying the (uniform) majorant condition*

$$\forall z \in S \; |f_n(z)| \leq x_n.$$

If $\sum_n x_n$ converges, then $\sum_n f_n(z)$ converges uniformly.

Proof. By the Comparison Test for number series, fixing $z \in S$, conclude that $\sum_n f_n(z)$ converges pointwise to some limit function $s(z)$, i.e.

$$\forall z \in S \; \sum_{k=1}^{n} f_k(z) \to s(z), \quad n \to \infty. \tag{9.2}$$

Fixing $\varepsilon > 0$ and using the convergence of $\sum_n x_n$, we can find N with

$$\sum_{n \geq N} x_n < \frac{\varepsilon}{2}.$$

Then for $m \geq N$, $z \in S$,

$$\left| \sum_{k=1}^{m+n} f_k(z) - \sum_{k=1}^{m} f_k(z) \right| = \left| \sum_{k=m+1}^{m+n} f_k(z) \right| \leq \sum_{k=m+1}^{m+n} x_k \leq \sum_{k=N}^{+\infty} x_k < \frac{\varepsilon}{2},$$

and taking the limit as $n \to \infty$ for each fixed $z \in S$, we obtain

$$\left| s(z) - \sum_{k=1}^{m} f_k(z) \right| \leq \frac{\varepsilon}{2} < \varepsilon, \quad m \geq N,$$

proving the desired uniform convergence, since N is independent of z. $\square$

Example 9.10. The series

$$\sum_{n\geq 1} \frac{z^n}{n^2}$$

converges uniformly in $S = \{z : |z| \leq 1\}$. Indeed, we have the uniform majorant estimate

$$|z| \leq 1 \implies \left|\frac{z^n}{n^2}\right| \leq \frac{1}{n^2}, \quad \sum_n \frac{1}{n^2} < +\infty,$$

and the uniform convergence on S follows from the Weierstrass M-test.

9.6. Continuity of uniform limits.

Theorem 9.11. *Let $f_n\colon S \to \mathbb{C}$ be a sequence of functions converging* uniformly *to a function $f\colon S \to \mathbb{C}$. If each f_n is continuous on S, so is the uniform limit f.*

Proof. We need to show for a fixed $z_0 \in S$ that f is continuous at z_0. Fixing $\varepsilon > 0$ and using the uniform convergence, we conclude there exists N such that

$$z \in S, n \geq N \implies |f_n(z) - f(z)| < \frac{\varepsilon}{3}. \tag{9.3}$$

By the continuity of f_N at z_0, there exists $\delta > 0$ with

$$z \in S, |z - z_0| < \delta \implies |f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}.$$

Then using (9.3) for z and $z = z_0$ and the triangle inequality,

$$|f(z) - f(z_0)| \leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f(z_0) - f_N(z_0)| < \varepsilon,$$

which holds for all $z \in S$ with $|z - z_0| < \delta$, proving the continuity of f at z_0 as desired. □

10. Holomorphic functions

Of fundamental importance is the notion of *differentiability* for functions $f(z)$ with both arguments and values in $\mathbb{C}$. Here there are two basic notions of differentiability: *complex and real differentiability.*

10.1. Complex-differentiable and holomorphic functions.

Definition 10.1 (complex differentiability)**.** A function $f\colon U \to \mathbb{C}$, where U is open, is called *complex-differentiable* or $\mathbb{C}$-*differentiable* at a point $z_0 \in U$ if there exists the limit

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}.$$

The limit $f'(z_0)$ is called the (complex) derivative of f at z_0.

Note that since U is open, every point z_0 is automatically a limit point of U, hence the limit makes sense.

Definition 10.2 (holomorphic functions)**.** A function f is *holomorphic* on an open set $U \subset \mathbb{C}$ if it is $\mathbb{C}$-differentiable at every point of U.

Example 10.3. Let $f(z) = c$ be a constant function for some $c \in \mathbb{C}$. Then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{c - c}{z - z_0} = 0$$

for all $z_0 \in \mathbb{C}$, hence f is $\mathbb{C}$-differentiable at every point with the derivative $f'(z_0) = 0$, and therefore is holomorphic on $\mathbb{C}$.

Example 10.4. Let $f(z) = z$ be the identity function. Then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = 1$$

for all $z_0 \in \mathbb{C}$, hence f is $\mathbb{C}$-differentiable at every point with the derivative $f'(z_0) = 1$, and therefore is holomorphic on $\mathbb{C}$.

Example 10.5. Let $f(z) = \overline{z}$ be the conjugation. Then f is not $\mathbb{C}$-differentiable at any point $z_0 \in \mathbb{C}$. To show this, write

$$z - z_0 = re^{i\theta}$$

in the polar form, and compute

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\overline{z} - \overline{z_0}}{z - z_0} = \frac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta},$$

which has no limit as $z \to z_0$. Indeed, suppose by contradiction that limit is some $L \in \mathbb{C}$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ with

$$|z - z_0| = r < \delta \implies |e^{-2i\theta} - L| < \varepsilon.$$

Choosing $r = \delta/2$, $\theta = 0$, we obtain

$$|1 - L| < \varepsilon,$$

and similarly for $r = \delta/2$, $\theta = \pi/2$, we obtain

$$|-1 - L| < \varepsilon,$$

which together with triangle inequality yields

$$2 = |(1 - L) - (-1 - L)| \leq |1 - L| + |-1 - L| < 2\varepsilon.$$

The latter is clearly impossible for $\varepsilon < 1$, a contradiction.

10.2. **Continuity of complex-differentiable functions.**

Lemma 10.6. *Let $f\colon U \to \mathbb{C}$ be $\mathbb{C}$-differentiable at z_0. Then f is continuous at z_0.*

Proof. Direct consequence of the computation:

$$\lim_{z\to z_0}(f(z)-f(z_0)) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\cdot\lim_{z\to z_0}(z-z_0) = f'(z_0)\cdot 0 = 0.$$

□

10.3. **Real differentiability.** We have identified complex numbers $z = x+iy$ with points (x,y) in the real plane $\mathbb{R}^2$. This identification turns every function $f\colon U\subset\mathbb{C}$ into $\mathbb{R}^2$-valued function of two real variables. Then Analysis of Several Real Variables provides us with different notion of differentiability that we call *real differentiability*:

Definition 10.7 (real differentiability: standard in Real Analysis)**.** A function $f\colon U\subset\mathbb{R}^m\to\mathbb{R}^n$, where U is open, is called *real-differentiable* or $\mathbb{R}$-*differentiable* at a point $x^0 = (x^0_1,\dots,x^0_m)\in U$ if there exists a real-linear map $L\colon\mathbb{R}^m\to\mathbb{R}^n$ with

$$\lim_{x\to x^0}\frac{\|f(x)-f(x^0)-L(x-x^0)\|}{\|x-x^0\|} = 0,\quad \|x\| = \sqrt{x_1^2+\ldots+x_m^2}. \tag{10.1}$$

Here "real-linear" means linear over real scalars, i.e.

$$L(tx+sy) = tL(x)+sL(y),\quad x,y\in\mathbb{R}^m,\ t,s\in\mathbb{R}.$$

The real-linear map $L(x-x^0)$ is then uniquely determined from (10.1) and is called the *differential* of f at x_0 satisfying the well-known formula

$$L(x-x^0) = df(x^0)(x-x^0) = \sum\frac{\partial f}{\partial x_j}(x^0)(x_j-x_j^0), \tag{10.2}$$

where $\frac{\partial f}{\partial x_j}$ is the *partial derivative* of f in the variable x_j:

$$\frac{\partial f}{\partial x_j}(x^0) := \lim_{t\to 0}\frac{f(x_1^0,\dots,x_{j-1}^0,x_j^0+t,x_{j+1}^0,\dots,x_m^0)-f(x^0)}{t}.$$

Lemma 10.8 ($\mathbb{C}$-differentiability $\implies$ $\mathbb{R}$-differentiability)**.** *Let $f\colon U\subset\mathbb{C}\to\mathbb{C}$ be* complex-differentiable *at z_0. Then it is also* real-differentiable *when viewed as map between points of the real plane $\mathbb{R}^2$.*

Proof. By definition, if f is *complex-differentiable* at z_0, there exists

$$f'(z_0) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} \iff \lim_{z\to z^0}\frac{f(z)-f(z^0)-f'(z_0)(z-z_0)}{z-z_0} = 0$$

and we conclude that f is *real-differentiable* whose differential $L = df(z_0)$ is given by the complex multiplication with derivative $f'(z_0)$:

$$L =: \mathbb{R}^2 \cong \mathbb{C} \to \mathbb{R}^2 \cong \mathbb{C}, \quad w \mapsto f'(z_0)w.$$

□

Not all $\mathbb{R}$-differentiable functions are $\mathbb{C}$-differentiable! Here is a simple counter-example:

Example 10.9. We have seen that the conjugate $f(z) = \overline{z}$ is not $\mathbb{C}$-differentiable at any z_0.

On the other hand, f is real-linear:

$$\overline{tz + sw} = t\overline{z} + s\overline{w}, \quad z, w \in \mathbb{C},\ t, s \in \mathbb{R},$$

hence its *real differentiability* at every point follows from the definition with $L = f$.

10.4. **Decomposition of the differential.** We write $\Delta x = x - x_0$, $\Delta y = y - y_0$ for the increments of each variable and $\Delta z = z - z_0$, where $z = x + iy$, $z_0 = x_0 + iy_0$. As *conjugation* is more convenient for computations (being a field automorphism), we perform the substitution

$$\Delta x = \frac{\Delta z + \overline{\Delta z}}{2}, \quad \Delta y = \frac{\Delta z - \overline{\Delta z}}{2i},$$

where $\Delta z = \Delta x + i\Delta y$, $\overline{\Delta z} = \Delta x - i\Delta y$.

Now compute the *differential* $L = df$ of f by the formula (10.2) (dropping the reference point z_0 for brevity):

$$df(\Delta x, \Delta y) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y = \frac{\partial f}{\partial x}\frac{\Delta z + \overline{\Delta z}}{2} + \frac{\partial f}{\partial y}\frac{\Delta z - \overline{\Delta z}}{2i}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)\Delta z + \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right)\overline{\Delta z} = \frac{\partial f}{\partial z}\Delta z + \frac{\partial f}{\partial \overline{z}}\overline{\Delta z},$$

where we group terms with Δz, $\overline{\Delta z}$, and introduce the *formal partial derivatives*:

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right). \tag{10.3}$$

Using (10.3) for $L = df$, we can rewrite (10.1) as

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0) - (\frac{\partial f}{\partial z}\Delta z + \frac{\partial f}{\partial \overline{z}}\overline{\Delta z})}{\Delta z} = 0. \tag{10.4}$$

10.5. **Comparison of real and complex differentiability.**

Theorem 10.10. *Let f be* real-differentiable *at z_0. Then the following are equivalent:*

(1) *f is* complex-differentiable *at z_0;*
(2) $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$.

If (1) (and hence (2)) holds, then $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

Proof. (2) $\implies$ (1): Assuming $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$, we have $df_{z_0}(\Delta x, \Delta y) = \frac{\partial f}{\partial z}(z_0)\Delta z$. Then (10.4) becomes

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z^0)}{\Delta z} - \frac{\partial f}{\partial z}(z_0) = 0$$

proving $\mathbb{C}$-differentiability (1) with $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

(1) $\implies$ (2): Rewriting (10.4) we have

$$\lim_{\Delta z \to 0} \left(\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - \frac{\partial f}{\partial z}(z_0) - \frac{\partial f}{\partial \bar{z}}(z_0)\frac{\overline{\Delta z}}{\Delta z} \right) = 0.$$

By (1) the 1st term has a limit, hence the 3rd term must also have a limit. However, we have seen that $\frac{\overline{\Delta z}}{\Delta z}$ has no limit as $\Delta z \to 0$. Then the only way the 3rd term has a limit is when $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$, i.e. when (2) holds. □

10.6. **Cauchy-Riemann equations.** Splitting $f = u + iv$ into real and imaginary parts, compute

$$2\frac{\partial f}{\partial \bar{z}} = \frac{\partial (u + iv)}{\partial x} - \frac{1}{i}\frac{\partial (u + iv)}{\partial y} = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right),$$

hence, identifying real and imaginary parts, we obtain

$$\frac{\partial f}{\partial \bar{z}} = 0 \iff \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases},$$

where on the right-hand side we obtain the *Cauchy-Riemann equations*. With that equivalence, Theorem 10.10 can be rewritten as

Corollary 10.11. *Let f be* real-differentiable *at z_0. Then the following are equivalent:*

(1) *f is* complex-differentiable *at z_0;*
(2) *the partial derivatives of $f = u + iv$ at z_0 satisfy the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$.*

If (1) (and hence (2)) holds, then $f'(z_0) = f_x(z_0) = u_x(z_0) + iv_x(z_0)$.

The last claim follows from the computation

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\lim_{\Delta x\to 0}\frac{f(z_0+\Delta x)-f(z_0)}{\Delta x}=f_x(z_0).$$

The following important result from Real Analysis provides a convenient way to show $\mathbb{R}$-differentiability:

Theorem 10.12 (Sufficient condition of $\mathbb{R}$-differentiability from Real Analysis)**.** *Let $U\subset\mathbb{R}^m$ is open and $f\colon U\to\mathbb{R}^m$ a function such that all partial derivatives*

$$\frac{\partial d}{\partial x_j},\quad j=1,\ldots,m,$$

exist and are continuous in U. The f is $\mathbb{R}$-differentiable at every point of U.

Example 10.13. The complex exponential function $f(z)=e^z$ was defined as

$$f(z)=e^x(\cos y+i\sin y),$$

so that we have for the real and imaginary parts

$$u(x,y)=e^x\cos y,\quad v(x,y)=e^x\sin y,$$

and we can directly verify the Cauchy-Riemann equations at every point:

$$u_x=e^x\cos y=v_y,\quad u_y=-e^x\sin y=-v_x. \tag{10.5}$$

Furthermore, both partial derivatives f_x, f_y exist and are continuous on $\mathbb{R}^2$ in view of (10.5). Hence $f(z)=e^z$ is real-differentiable at every point of $\mathbb{C}$ by Theorem 10.12. Then by Corollary 10.11, $f(z)=e^z$ is $\mathbb{C}$-differentiable at every point and therefore is holomorphic on $\mathbb{C}$.

Furthermore, by Corollary 10.11,

$$(e^z)'=\frac{\partial}{\partial x}e^x(\cos y+i\sin y)=e^x(\cos y+i\sin y)=e^z.$$

10.7. **Algebraic properties of $\mathbb{C}$-differentiability.** Algebraic properties of $\mathbb{C}$-differentiability are analogous to those of differentiability of functions of one real variable with the same proofs:

Theorem 10.14. *Let $U\subset\mathbb{C}$ be open and $f,g\colon U\to\mathbb{C}$ be $\mathbb{C}$-differentiable at $z_0\in U$. Then*

(1) *$f+g$ is $\mathbb{C}$-differentiable at z_0 with*

$$(f+g)'(z_0)=f'(z_0)+g'(z_0);$$

(2) *fg is $\mathbb{C}$-differentiable at z_0 with*

$$(fg)'(z_0)=f'(z_0)g(z_0)+f(z_0)g'(z_0);$$

(3) *if* $g(z) \neq 0$ *for* $z \in U$*, the ratio* $\frac{f}{g}$ *is* $\mathbb{C}$*-differentiable at* z_0 *with*

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}.$$

Example 10.15. We have seen that the constant and the identity functions are holomorphic on $\mathbb{C}$. Taking products and sums, we conclude that any polynomial function $P(z) = a_n z^n + \ldots + a_0$ is $\mathbb{C}$-differentiable at every $z_0 \in \mathbb{C}$, hence is holomorphic on $\mathbb{C}$. Taking ratios of polynomials, we conclude that any rational function $f(z) = \frac{P(z)}{Q(z)}$ is holomorphic on the open set $U := \{z \in \mathbb{C} : Q(z) \neq 0\}$.

10.8. **Compositions and inverses of $\mathbb{C}$-differentiable functions.**

Theorem 10.16. *Let* $U, V \subset \mathbb{C}$ *be open and*

$$f\colon U \to \mathbb{C}, \quad g\colon V \to \mathbb{C}, \quad f(U) \subset V,$$

so that the composition $g \circ f\colon U \to \mathbb{C}$ *is defined. If* f *and* g *are* $\mathbb{C}$*-differentiable at* z_0 *and* $w_0 := f(z_0)$ *respectively, then* h *is* $\mathbb{C}$*-differentiable at* z_0 *with derivative given by the* chain rule

$$(g \circ f)'(z_0) = g'(w_0)f'(z_0).$$

Proof. We have

$$\frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} = G(f(z))\frac{f(z) - f(z_0)}{z - z_0}, \quad z \neq z_0, \tag{10.6}$$

where

$$G(w) := \begin{cases} \frac{g(w)-g(w_0)}{w-w_0} & w \neq w_0 \\ g'(w_0) & w = w_0 \end{cases}.$$

Since g is $\mathbb{C}$-differentiable at w_0, G is continuous at w_0 and hence $G(f(z))$ is continuous at z_0 as composition of continuous functions and

$$\lim_{z \to z_0} G(f(z)) = g'(w_0).$$

Since f is $\mathbb{C}$-differentiable at z_0, (10.6) yields

$$\lim_{z \to z_0} \frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} = g'(w_0)f'(z_0)$$

as desired. □

Theorem 10.17. *Let* $U, V \subset \mathbb{C}$ *be open subsets,* $f\colon U \to \mathbb{C}$ *a function, and* $f^{-1}\colon V \to U$ *a right inverse, i.e.*

$$f(f^{-1}(w)) = w, \quad w \in V. \tag{10.7}$$

Assume that

(1) *f^{-1} is continuous at $w_0 \in V$.*
(2) *f is $\mathbb{C}$-differentiable at $z_0 = f^{-1}(w_0) \in U$;*
(3) *$f'(z_0) \neq 0$;*

Then f^{-1} is $\mathbb{C}$-differentiable at w_0 with derivative

$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)} = \frac{1}{f'(f^{-1}(w_0))}.$$

Proof. For $w \neq w_0 \in V$ set $z = f^{-1}(w)$. Then $f(z) = w$, $f(z_0) = w_0$ by (10.7) and

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)}. \tag{10.8}$$

By (2) and (3), we have

$$\lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)}. \tag{10.9}$$

To show that the left-hand side of (10.8) converges to the same limit as $w \to w_0$, we need to use (1). Indeed, (10.9) means for every $\varepsilon > 0$ there exists $\eta > 0$ with

$$z \neq z_0, |z - z_0| < \eta \implies \left| \frac{z - z_0}{f(z) - f(z_0)} - \frac{1}{f'(z_0)} \right| < \varepsilon.$$

Since $f(f^{-1}(w)) = f(z) = w$, $w \neq w_0$ implies $z \neq z_0$. Since f^{-1} is continuous at w_0 and there exists $\delta > 0$ with

$$w \neq w_0, |w - w_0| < \delta \implies z \neq z_0, |z - z_0| < \eta,$$

and therefore

$$w \neq w_0, |w - w_0| < \delta \implies \left| \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} - \frac{1}{f'(z_0)} \right| < \varepsilon$$

proving the desired conclusion. □

Example 10.18. Let $V \subset \mathbb{C} \setminus \{0\}$ be open and $g \colon V \to \mathbb{C}$ a branch of the logarithm $\log(w)$. Then

$$f(g(w)) = w,$$

holds for

$$f(z) = e^z, \quad z \in U = \mathbb{C},$$

i.e. $g = f^{-1}$ is a right inverse in the sense that $f \circ f^{-1} = \mathsf{id}$. Then by Theorem 10.17, the branch $g = f^{-1}$ is $\mathbb{C}$-differentiable at every $w_0 \in V$ with

$$g'(w_0) = \frac{1}{f'(g(w_0))} = \frac{1}{f(g(w_0))} = \frac{1}{w_0}$$

since $f'(z) = (e^z)' = e^z = f(z)$ and $f(g(w)) = w$.

For example, the principal logarithm $\mathsf{Log}\, z$ is a branch over the open set $V := \mathbb{C} \setminus (-\infty, 0]$. (Note that we defined $\mathsf{Log}\, z$ for all $z \neq 0$ but it is continuous, hence a branch, only over V.) Then we have shown above that $\mathsf{Log}\, z$ is holomorphic on V with

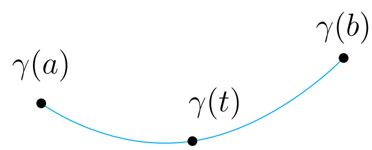
$$(\mathsf{Log})'(w) = \frac{1}{w}, \quad w \in V.$$

11. Paths in complex plane

Definition 11.1. (1) A *path* in a subset $S \subset \mathbb{C}$ is a *continuous* map $\gamma\colon [a, b] \to S$, where $[a, b] \subset \mathbb{R}$ is an interval with $a < b$.

(2) The point $\gamma(a)$ is called the *beginning point* and $\gamma(b)$ the *end point* of γ.

(3) A path γ is called *closed* whenever it has the same beginning and the end point, i.e. $\gamma(a) = \gamma(b)$.



Note that we can write

$$\gamma(t) = \alpha(t) + i\beta(t), \quad \alpha(t) = \mathsf{Re}\, \gamma(t), \quad \beta(t) = \mathsf{Im}\, g(t).$$

Lemma 11.2. *A function $\gamma(t) = \alpha(t) + i\beta(t)$, $t \in [a, b]$, is* continuous *if and only if both $\alpha(t)$ and $\beta(t)$ are continuous.*

Proof. If $\gamma(t)$ is continuous, and since $\mathsf{Re}\, w$ and $\mathsf{Im}\, w$ are continuous in w, their compositions $\alpha(t)$ and $\beta(t)$ are continuous.

Vice versa, if $\alpha(t)$ and $\beta(t)$ are continuous, then $\gamma(t)$ is continuous as a sum of products of continuous functions. □

11.1. Differentiability of paths.

Definition 11.3. (1) A path $\gamma\colon [a, b] \to \mathbb{C}$ is *differentiable* at $t_0 \in [a, b]$ if there exists

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} =: \gamma'(t_0) \in \mathbb{C},$$

which we call the derivative of γ at t_0.

(2) A path $\gamma\colon [a, b] \to \mathbb{C}$ is *piecewise continuously differentiable* of *piecewise C^1* if there exists a partition

$$a = t_0 < t_1 < \ldots < t_n = b$$

such that each restriction path

$$\gamma_j := \gamma|_{[t_{j-1},t_j]} \colon [t_{j-1}, t_j] \to \mathbb{C}$$

is differentiable at every $t \in [t_{j-1}, t_j]$ and its derivative $\gamma_j'(t)$ is continuous on $[t_{j-1}, t_j]$.

Taking real and imaginary parts, we obtain:

Lemma 11.4. *A path*

$$\gamma(t) = \alpha(t) + i\beta(t)$$

is differentiable at t_0 (resp. piecewise continuously differentiable) if and only if both $\alpha(t)$ and $\beta(t)$ are differentiable at t_0 (resp. piecewise continuously differentiable). If γ is differentiable at t_0, its derivative is

$$\gamma'(t_0) = \alpha'(t_0) + i\beta'(t_0).$$

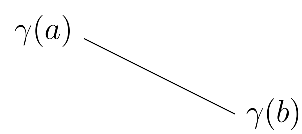
If both $\alpha(t)$ and $\beta(t)$ are piecewise continuously differentiable with respective partiions $a = t_0 < \ldots < t_n = b$ and $a = s_0 < \ldots < s_m = b$, we can use the refined partition given by the set of all t_j and s_k to show that $\gamma(t)$ is piecewise continuously differentiable.

11.2. **Affine and piecewise affine paths.**

Example 11.5 (Affine path). For $A, B \in \mathbb{C}$, let

$$\gamma(t) := At + B, \quad t \in [a, b],$$

whose image $\gamma([a, b])$ is the line segment between $\gamma(a)$ and $\gamma(b)$:



Clearly γ is continuously differentiable as a sum of products of continuously differentiable functions.

Example 11.6 (Piecewise affine path). More generally, consider a partition

$$a = t_0 < t_1 < \ldots < t_n = b$$

and for each $j = 1, \ldots, n$ an affine linear path

$$\gamma_j(t) = A_j t + B_j, \quad t \in [t_{j-1}, t_j],$$

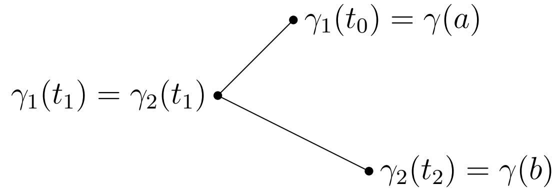
such that

$$\gamma_1(t_1) = \gamma_2(t_1), \gamma_2(t_2) = \gamma_3(t_2), \ldots, \gamma_{n-1}(t_{n-1}) = \gamma_n(t_{n-1}).$$

Then combining all γ_j, we define the *piecewise affine* path $\gamma\colon [a,b] \to \mathbb{C}$ by

$$\gamma(t) := \gamma_j(t) \text{ for } t_{j-1} \le t \le t_j,$$

which is piecewise C^1:



11.3. **Integrals of $\mathbb{C}$-valued functions.** The Riemann integral of $\mathbb{C}$-valued functions $\varphi\colon [a,b] \to \mathbb{C}$ is defined analogously to the Riemann integral of $\mathbb{R}$-valued functions as limit of integral sums

$$\int_a^b \varphi(t)dt = \lim_{\max_j (t_j - t_{j-1}) \to 0} \sum_j \varphi(s_j)(t_j - t_{j-1}), \tag{11.1}$$

where $a = t_0 < \ldots < t_n = b$ is a partition and $s_j \in [t_{j-1}, t_j]$.

As for the derivative, separating real and imaginary parts we obtain:

Lemma 11.7. *A function $\varphi = \alpha + i\beta\colon [a,b] \to \mathbb{C}$ is integrable if and only if both real and imaginary parts α, β are integrable. If $\varphi = \alpha + i\beta$ is integrable,*

$$\int_a^b \varphi(t)dt = \int_a^b \alpha(t)dt + i\int_a^b \beta(t)dt.$$

Taking absolute values in (11.1) and using the triangle inequality, we obtain:

Lemma 11.8 (Basic estimate)**.**

$$\left|\int_a^b \varphi(t)dt\right| \le \sup_{t\in[a,b]} |\varphi(t)|(b-a).$$

Again, separating al and imaginary parts, the Fundamental theorem of calculus can be extended to $\mathbb{C}$-valued functions:

Theorem 11.9 (Fundamental theorem of calculus)**.** *Let $\varphi\colon [a,b] \to \mathbb{C}$ be continuously differentiable. Then*

$$\int_a^b \varphi'(t)dt = \varphi(b) - \varphi(a).$$

11.4. **Integral of a function along a path.** If $f\colon S \to \mathbb{C}$ is a continuous function and $\gamma\colon [a,b] \to S$ is a continuously differentiable path, define the integral of f along γ by

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt. \tag{11.2}$$

Since the composition $f(\gamma(t))$ and the derivative $\gamma'(t)$ are continuous, the integral in (11.2) always exists (as the Riemann integral).

More abstractly, $\int_\gamma f(z)dz$ can be defined for more general γ and f as the limit of the integral sums

$$\sum_j f(\gamma(s_j))(\gamma(t_j) - \gamma(t_{j-1})), \tag{11.3}$$

as $\max_j(t_j - t_{j-1}) \to 0$, where $a = t_0 < \ldots < t_n = b$ is a partition and $s_j \in [t_{j-1}, t_j]$.

Lemma 11.10. *Let $f\colon S \to \mathbb{C}$ be a continuous function and $\gamma\colon [a,b] \to S$ a continuously differentiable path. Then the integral sum* (11.3) *converges to* (11.2) *as* $\max_j(t_j - t_{j-1}) \to 0$.

Proof. By Theorem 11.9,

$$\gamma(t_j) - \gamma(t_{j-1}) = \int_{t_{j-1}}^{t_j} \gamma'(t)dt.$$

We need to estimate

$$\begin{aligned}
\Delta &:= \left| \sum_j f(\gamma(s_j))(\gamma(t_j) - \gamma(t_{j-1})) - \int_a^b f(\gamma(t))\gamma'(t)dt \right| \\
&= \left| \sum_j f(\gamma(s_j))(\gamma(t_j) - \gamma(t_{j-1})) - \sum_j \int_{t_{j-1}}^{t_j} f(\gamma(t))\gamma'(t)dt \right| \\
&\le \sum_j \left| f(\gamma(s_j))(\gamma(t_j) - \gamma(t_{j-1})) - \int_{t_{j-1}}^{t_j} f(\gamma(t))\gamma'(t)dt \right| \\
&\le \sum_j \left| f(\gamma(s_j)) \int_{t_{j-1}}^{t_j} \gamma'(t)dt - \int_{t_{j-1}}^{t_j} f(\gamma(t))\gamma'(t)dt \right| \\
&\le \sum_j \left| \int_{t_{j-1}}^{t_j} \Big(f(\gamma(s_j))\gamma'(t) - f(\gamma(t))\gamma'(t) \Big) dt \right| \\
&\le \sum_j \sup_{t \in [t_{j-1}, t_j]} |f(\gamma(s_j)) - f(\gamma(t))| |\gamma'(t)| (t_j - t_{j-1}),
\end{aligned}$$

where we used the basic estimate in Lemma 11.8. We next use the compactness of interval $[a,b]$, the *boundedness* of the continuous function $|\gamma'(t)|$ on the compact set $[a,b]$:

$$|\gamma'(t)| \le M, \quad t \in [a,b],$$

for some $M > 0$, and the *uniform continuity* of the continuous function $f(\gamma(t))$ on $[a,b]$ by Theorem 8.25, i.e. given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|s_j - t| < \delta \implies |f(\gamma(s_j)) - f(\gamma(t))| < \varepsilon.$$

Then $\max_j(t_j - t_{j-1}) < \delta$ implies

$$\Delta \le \sum_j \varepsilon M(t_j - t_{j-1}) = \varepsilon M(b-a),$$

which proves $\Delta \to 0$ as $\max_j(t_j - t_{j-1}) \to 0$ as desired. □

11.5. **Piecewise C^1 paths.** For our purposes, it will suffice to consider only integrals of continuous functions along C^1 and piecewise C^1 paths. In order to define it for piecewise C^1 paths, observe that (11.2) satisfies the important *additivity property*:

Lemma 11.11. *Let $f\colon S \to \mathbb{C}$ be a continuous function and $\gamma\colon [a,b] \to S$ a continuously differentiable path. Then for every partition $a = t_0 < \ldots < t_n = b$, we have*

$$\int_\gamma f(z)dz = \sum_j \int_{\gamma_j} f(z)dz, \quad \gamma_j := \gamma|_{[t_{j-1},t_j]}.$$

Proof. The proof follows from (11.2) and additivity of the Riemann integral

$$\int_a^b g(t)dt = \sum_j \int_{t_{j-1}}^{t_j} g(t)dt,$$

where $g(t) = f(\gamma(t))\gamma'(t)$. □

Definition 11.12. Let $f\colon S \to \mathbb{C}$ be a continuous function and $\gamma\colon [a,b] \to S$ a piecewise C^1 path. Consider any partition

$$a = t_0 < \ldots < t_n = b$$

such that each $\gamma_j := \gamma|_{[t_{j-1},t_j]}$ is C^1. Then the integral of f along γ is defined by

$$\int_\gamma f(z)dz := \sum_{j=1}^n \int_{\gamma_j} f(z)dz = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} f(\gamma_j(t))\gamma_j'(t)dt. \tag{11.4}$$

Note that we need to show the integral is well-defined by proving that (11.4) is independent of the choice of the partition by t_j. To show this, consider another partition by points $s_k \in [a, b]$ such that each restriction $\gamma|_{[s_{k-1},s_k]}$ is C^1 and take the refinement partition defined by the union of all t_j and s_k. Then it follows from Lemma 11.11 that the sum in (11.4) for each of the partitions (t_j) and (s_k) equals the corresponding sum for the refinement. Hence all sums are equal as desired.

We can extend Lemma 11.10 to piecewise C^1 paths:

Lemma 11.13. *Let $f : S \to \mathbb{C}$ be a continuous function and $\gamma : [a, b] \to S$ a piecewise C^1 path. Then*

$$\int_\gamma f(z)dz = \lim_{\max_k (t_k - t_{k-1}) \to 0} \sum_k f(\gamma(s_k))(\gamma(t_k) - \gamma(t_{k-1})),$$

where $a = t_0 < \ldots < t_n = b$ is a partition and $s_k \in [t_{k-1}, t_k]$.

Proof. Fix a partition by t_j^0 as in Definition 11.12. The each integral $\int_{\gamma_j} f(z)dz$ is the limit of its integral sums. Taking their sum for all $k = 0, \ldots, n$, we obtain we obtain $\int_\gamma f(z)dz$ as limit of integral sums over partitions that include all t^0. A more general partition can be refined to one including all t^0 and can be shown to be arbitrary close to its refinement as in the proof of Lemma 11.10. □

11.6. **Linearity of the integral.** Another important property of the integral the linearity:

Lemma 11.14. *Let $f, g : S \to \mathbb{C}$ be piecewise continuous functions, $\gamma : [a, b] \to S$ a piecewise C^1 path, and $A, B \in \mathbb{C}$. Then*

$$\int_\gamma (Af(z) + Bg(z))dz = A \int_\gamma f(z)dz + B \int_\gamma g(z)dz \tag{11.5}$$

Proof. Using a partition as in (11.4), it suffices to show (11.5) for γ continuously differentiable, which follows from the linearity of integral sums

$$\sum_j (Af(\gamma(s_j) + Bg(\gamma(s_j))(\gamma(t_j) - \gamma(t_{j-1}))$$

$$= A \sum_j f(\gamma(s_j)(\gamma(t_j) - \gamma(t_{j-1})) + B \sum_j g(\gamma(s_j))(\gamma(t_j) - \gamma(t_{j-1}))$$

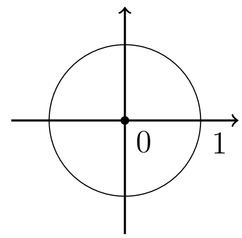
and Lemma 11.10. □

11.7. **Example: integral along the unit circle.**

Example 11.15. Consider the function $f(z) = \frac{1}{z}$ defined on $U = \mathbb{C} \setminus \{0\}$ and the closed path

$$\gamma(t) = e^{it}, \quad t \in [0, 2\pi],$$

parametrizing the unit circle:



Since e^{iz} is $\mathbb{C}$-differentiable with $(e^{iz})' = ie^{iz}$ by the chain rule, we compute the derivative of γ by restricting the limit to the real axis

$$(e^{it})'(t_0) = \lim_{t \to t_0} \frac{e^{it} - e^{it_0}}{t - t_0} = \lim_{z \to t_0} \frac{e^{iz} - e^{it_0}}{z - t_0} = ie^{it_0},$$

and the integral of f along γ as

$$\int_\gamma \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} (e^{it})' dt = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = \int_0^{2\pi} i dt = 2\pi i.$$

11.8. **Antiderivatives and integrals.**

Definition 11.16. Let $U \subset \mathbb{C}$ be open. An *antiderivative* of a function $f \colon U \to \mathbb{C}$ is any holomorphic function $F \colon U \to \mathbb{C}$ with

$$F'(z) = f(z), \quad z \in U.$$

The following analogue of the fundamental theorem of calculus can be a convenient way to compute integrals:

Theorem 11.17. *Let $U \subset \mathbb{C}$ be open, and $f \colon U \to \mathbb{C}$ a continuous function with an antiderivative F. Then for every piecewise C^1 path $\gamma \colon [a, b] \to U$,*

$$\int_\gamma f(z) dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. We first assume γ is C^1. By the chain rule, for $\varphi(t) := F(\gamma(t))$,

$$\varphi'(t) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t).$$

Then by Theorem 11.9,

$$\int_\gamma f(z) dz = \int_a^b f(\gamma(t))\gamma'(t) dt = \int_a^b \varphi'(t) dt = \varphi(b) - \varphi(a) = F(\gamma(b)) - F(\gamma(a)),$$

proving the claim for any γ of class C^1.

More generally, for γ piecewise C^1, consider any partition $a = t_0 < \ldots < t_n = b$ as in Defintion 11.12, so that

$$\int_\gamma f(z)dz = \sum_j \int_{\gamma_j} f(z)dz,$$

where each γ_j is C^1. Then we have proved that

$$\int_{\gamma_j} f(z)dz = F(\gamma(t_j)) - F(\gamma(t_{j-1})),$$

and taking the sum over $j = 1, \ldots, n$, we obtain the claim. □

Example 11.18. The power function $f(z) = z^n$ has the antiderivative $F(z) = \frac{z^{n+1}}{n+1}$ in the entire complex plane $U = \mathbb{C}$. Then for every piecewise C^1 path $\gamma\colon [a,b] \to U$,

$$\int_\gamma z^n dz = \frac{(\gamma(b))^{n+1} - (\gamma(a))^{n+1}}{n+1}.$$

11.9. **Reparametrizations of paths.**

Lemma 11.19. *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise C^1 path and*

$$\psi : [\alpha, \beta] \to [a,b]$$

an increasing piecewise C^1 bijection. Then

$$\int_{\gamma\circ\psi} f(z)dz = \int_\gamma f(z)dz,$$

for any continuous function f.

Proof. Taking a partition of $[\alpha,\beta]$ and corresponding partition of $[a,b]$, the statement is reduced to the case when f and ψ are C^1. Then

$$(\gamma\circ\psi)'(t) = \gamma'(\psi(t))\psi'(t)$$

by the chain rule and

$$\int_{\gamma\circ\psi} f(z)dz = \int_\alpha^\beta f(\gamma(\psi(t)))\gamma'(\psi(t))\psi'(t)dt$$

$$= \int_a^b f(\gamma(s))\gamma'(s)ds = \int_\gamma f(z)dz$$

by the change of variable in the integral. □

Lemma 11.20 (orientation reversing reparametrization)**.** *Let* $\gamma\colon [a,b] \to \mathbb{C}$ *be a piecewise* C^1 *path and*

$$\psi\colon [\alpha,\beta] \to [a,b]$$

a decreasing piecewise C^1 *bijection. Then*

$$\int_{\gamma\circ\psi} f(z)dz = -\int_\gamma f(z)dz,$$

for any continuous function f*.*

Proof. Consider the increasing bijection

$$\varphi\colon [-\beta,-\alpha] \to [a,b], \quad t \mapsto \psi(-t).$$

Then by Lemma 11.19,

$$\int_{\gamma\circ\varphi} f(z)dz = \int_\gamma f(z)dz.$$

Taking a partition

$$-\beta = t_0 < \ldots < t_n = -\alpha, \quad s_j \in [t_{j-1}, t_j],$$

we obtain

$$\sum f(\gamma(\varphi(s_j)))(\gamma(\varphi(t_j)) - \gamma(\varphi(t_{j-1}))$$
$$= -\sum f(\gamma(\psi(-s_j)))(\gamma(\psi(-t_{j-1}) - \gamma(\psi(-t_j)))$$

for the corresponding integral sums, proving

$$\int_{\gamma\circ\varphi} f(z)dz = -\int_{\gamma\circ\psi} f(z)dz,$$

and the conclusion follows. □

11.10. **Length of path and basic estimate of integral.**

Lemma 11.21. *Let* $\varphi\colon [a,b] \to \mathbb{C}$ *be a path. Then*

$$\left|\int_a^b \varphi(t)dt\right| \le \int_a^b |\varphi(t)|dt.$$

Proof. Write

$$\int_a^b \varphi(t)dt = re^{i\theta}$$

in the polar form. Then

$$\left|\int_a^b \varphi(t)dt\right| = r = e^{-i\theta}\int_a^b \varphi(t)dt = \int_a^b e^{-i\theta}\varphi(t)dt.$$

The right-hand side is a real number, hence it is equal to

$$\int_a^b \mathsf{Re}\,(e^{-i\theta}\varphi(t))dt \le \int_a^b |e^{-i\theta}\varphi(t)|dt = \int_a^b |\varphi(t)|dt.$$

□

Definition 11.22. The *length* of a C^1 path $\gamma\colon [a,b] \to \mathbb{C}$ is

$$L(\gamma) := \int_a^b |\gamma'(t)|dt.$$

The length of a piecewise C^1 path is the sum of the lengths of its C^1 pieces:

$$L(\gamma) := \sum L(\gamma_j),$$

where each $\gamma_j = \gamma|_{[t_{j-1},t_j]}$ is C^1 for a partition $a = t_0 < \ldots < t_n = b$.

Theorem 11.23 (Basic estimate for integral along path)**.** *Let $f\colon S \to \mathbb{C}$ be continuous and $\gamma\colon [a,b] \to S$ a piecewise C^1 path. Then*

$$\left|\int_\gamma f(z)dz\right| \le \sup_{z\in\gamma([a,b])} |f(z)|L(\gamma).$$

Proof. If γ is C^1, we have

$$\left|\int_\gamma f(z)dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \le \int_a^b |f(\gamma(t))\gamma'(t)|dt$$

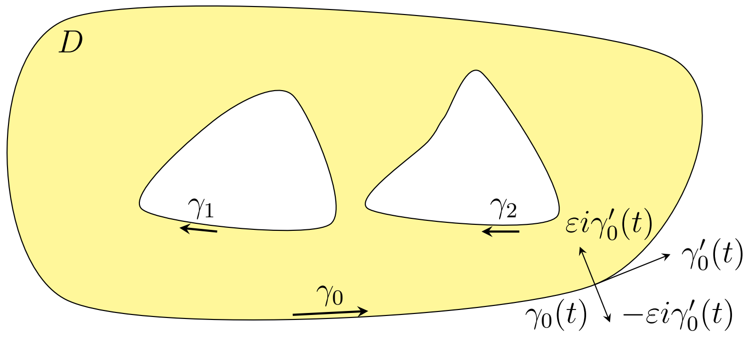
$$\le \int_a^b \Big(\sup_{z\in\gamma([a,b])} |f(z)|\Big)|\gamma'(t)|dt \le \sup_{z\in\gamma([a,b])} |f(z)| \int_a^b |\gamma'(t)|dt$$

in view of Lemma 11.21. Then the right-hand side equals $\sup_{z\in\gamma([a,b])} |f(z)|L(\gamma)$ as desired.

More generally, if γ is piecewise C^1, let γ_j be as in Definition 11.22. Then the statement holds for each γ_j and the conclusion follows by taking the sum for all j. □

12. Cauchy's Theorem

12.1. **Oriented boundary.** In the following we shall consider open regions D whose boundary can be represented as unions of images of piecewise C^1 paths:



We have seen that the integral along path $\gamma\colon [a,b] \to \mathbb{C}$ does not change under increasing reparametrizations $[\alpha,\beta] \to [a,b]$, but changes sign under decreasing reparametrizations. Hence, need to specify an *orientation* for the paths representing the boundary. Intuitively, our convention will be that the region D remains on the left of each boundary path.

In the above illustration, we use the path $\gamma_0, \gamma_1, \gamma_2$ to represent the oriented boundary of D and define the integral

$$\int_{\partial D} f(z)dz := \sum_{k=0}^{2} \int_{\gamma_k} f(z)dz$$

for any continuous function f defined on the boundary ∂D.

Definition 12.1. Let $U \subset \mathbb{C}$ be open and $\gamma\colon [a,b] \to \mathbb{C}$. We say that

(1) D is *on the left* of a piecewise C^1 path with $\gamma([a,b]) \subset \partial D$, if for every $t \in (a,b)$ where $\gamma'(t)$ exists,

$$\gamma'(t) \neq 0, \quad \gamma(t) + \varepsilon i\gamma'(t) \in D, \quad \gamma(t) - \varepsilon i\gamma'(t) \in \mathbb{C} \setminus \overline{D},$$

holds for every $\varepsilon > 0$ sufficiently small.

(2) a finite union of closed paths $\gamma_j\colon [a_j,b_j] \to \mathbb{C}$ represents the *oriented boundary* of D, if D is on the left of each γ_j, and

$$\partial D = \cup_j \gamma_j([a_j,b_j]), \quad \gamma_j([a_j,b_j]) \cap \gamma_k([a_k,b_k]) = \emptyset.$$

(3) if a finite union of γ_j represents the oriented boundary of D, we define the integral

$$\int_{\partial D} f(z)dz := \sum_j \int_{\gamma_j} f(z)dz,$$

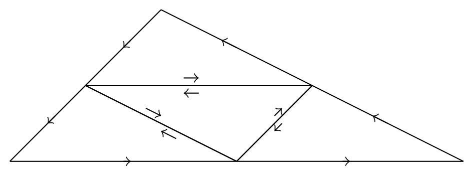
where f is any continuous function on ∂D.

12.2. **Cauchy-Goursat Theorem for triangle.**

Theorem 12.2 (Goursat)**.** *Let $U \subset \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ be holomorphic, and T a triangle whose closure $\overline{T}$ is contained in U. Then*

$$\int_{\partial T} f(z)dz = 0,$$

where ∂T is the oriented boundary.



Proof. We denote our original triangle by $T_0 = T$ and assume by contradiction

$$\left| \int_{\partial T} f(z)dz \right| =: M > 0. \tag{12.1}$$

Then bisect each edge of the triangle and create four new triangles, $\Delta_{1,2,3,4}$, see the above illustration. With our choice of orientation, the edges in the interior have opposing orientations as oriented boundary for the triangles on each side. Because of this, the contributions from integrals along these edges cancel in the sum of $\int_{\partial T_n} f(z)dz$, and

$$\int_{\partial T_0} f(z)dz = \sum_{n=1}^{4} \int_{\partial \Delta_n} f(z)dz.$$

Then there is at least one triangle T_1 among $\Delta_{1,2,3,4}$ such that

$$\left| \int_{\partial T_1} f(z)dz \right| \geq \frac{M}{4}.$$

We keep repeating the same process to obtain a sequence of triangles

$$T_0 \supset T_1 \supset \ldots \supset T_n \ldots$$

with

$$\left| \int_{\partial T_n} f(z)dz \right| \geq \frac{M}{4^n}. \tag{12.2}$$

Next we choose a sequence $z_n \in T_n$ for $n \geq 1$, and claim that (z_n) is Cachy. Writing $L(\partial T_n)$ for the perimeter of T_n, we have

$$L(\partial T_n) = \frac{L(\partial T)}{2^n}.$$

Then for $m \geq n$, we have $T_m \subset T_n$ and hence

$$z_m, z_n \in T_n \implies |z_m - z_n| \leq L(\partial T_n) = \frac{L(\partial T)}{2^n}.$$

To show that (z_n) is a Cauchy sequence, for every $\varepsilon > 0$, choose N with $\frac{L(\partial T)}{2^n} < \varepsilon$ for $n \geq N$. By Cauchy's criterion, z_n converges to some $z_0 \in \overline{T}$. Since $z_m \in T_n$ for all $m \geq n$, we also must have $z_0 \in \overline{T}_n$.

We next use $\mathbb{C}$-differentiability of f at z_0,

$$\lim_{z \to z_0} R(z) = 0, \quad R(z) := \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0}.$$

Then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + R(z)(z - z_0)$$

(12.3)

$$\implies \int_{\partial T_n} f(z)dz = \int_{\partial T_n} (f(z_0) + f'(z_0)(z - z_0))dz + \int_{\partial T_n} R(z)(z - z_0)dz.$$

To compute the first integral on the right, observe that the function $g(z) = f(z_0) + f'(z_0)(z - z_0)$ is affine and therefore has an antiderivative G, e.g.

$$G(z) = f(z_0)(z - z_0) + \frac{f'(z_0)}{2}(z - z_0)^2.$$

Since ∂T_n can be represented by a closed path $\gamma_n \colon [a_n, b_n] \to \mathbb{C}$, i.e. with $\gamma_n(a_n) = \gamma_n(b_n)$, we have

$$\int_{\partial T_n} g(z)dz = \int_{\gamma_n} g(z)dz = G(\gamma_n(b_n)) - G(\gamma_n(a_n)) = 0$$

for the 1st integral in (12.3). On the other hand, we use the basic estimate in Theorem 11.23 for the 2nd integral:

$$\left| \int_{\partial T_n} R(z)(z - z_0)dz \right| \leq \sup_{z \in \partial T_n} |R(z)(z - z_0)| L(\partial T_n). \tag{12.4}$$

Since $R(z) \to 0$, $z \to z_0$, for every $\varepsilon > 0$ there exists N such that

$$n \geq N \implies \sup_{z \in \partial T_n} |R(z)| \leq \varepsilon.$$

Since $z_0 \in \overline{T}_n$ for every n, we have

$$z \in \partial T_n \implies |z - z_0| \leq L(\partial T_n).$$

Hence

$$\sup_{z \in \partial T_n} |R(z)(z - z_0)| L(\partial T_n) \leq \varepsilon L(\partial T_n)^2 = \frac{\varepsilon L(\partial T)^2}{4^n}.$$

In view of (12.3) and (12.4), this yields

$$\left|\int_{\partial T_n} f(z)dz\right| \leq \frac{\varepsilon L(\partial T)^2}{4^n}.$$

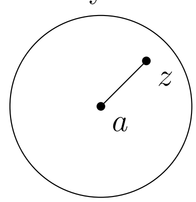
Finally, for ε sufficiently small, we obtain a contradiction with (12.2), completing the proof. □

12.3. **Cauchy's Theorem for star-shaped sets.** For fixed $z, w \in \mathbb{C}$, denote by $[z, w]$ the line segment connecting z and w, i.e.

$$[z, w] := \{(1-t)z + tw : t \in [0, 1]\}.$$

Definition 12.3. A subset $U \subset \mathbb{C}$ is called *start-shaped* if there exists $z_0 \in U$ such that $[z, z_0] \subset U$ for all $z \in U$.

Example 12.4. Any disc $\Delta_r(a) = \{z : |z - a| < r\}$ is star-shaped with $z_0 = a$, since $[a, z] \subset \Delta_r(a)$ for every $z \in \Delta_r(a)$.



Lemma 12.5. *Let $U \subset \mathbb{C}$ be open and star-shaped and $f : U \to \mathbb{C}$ a continuous function. Suppose*

$$\int_{\partial T} f(z)dz = 0 \tag{12.5}$$

for all triangles T with $\overline{T} \subset U$. Then f has an antiderivative in U, i.e. a holomorphic function $F : U \to \mathbb{C}$ with $F'(z) = f(z)$ for all $z \in U$. In fact, we can take

$$F(z) = \int_{[z_0,z]} f(w)dw, \tag{12.6}$$

where $z_0 \in U$ satisfies $[z_0, z] \in U$ for all $z \in U$ as per Definition 12.3.

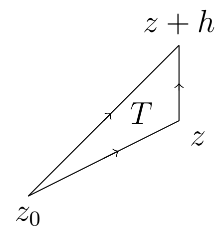
Proof. Fix $z \in U$ and h small enough so that $z + h \in U$. We need to show that $F(z)$ is $\mathbb{C}$-differentiable at z and that its derivative equals $f(z)$. In other words, we need to show that

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

From the definition (12.6) of F, we find

$$F(z+h) - F(z) = \int_{[z_0,z+h]} f(w)dw - \int_{[z_0,z]} f(w)dw,$$

where both integrals are taken over the line segments. If we now consider the third integral $\int_{[z,z+h]} f(w)\mathrm{d}w$, then these three line segments form a triangle T:



By our assumption (12.5),

(12.7)
$$0 = \int_{\partial T} f(w)dw = \int_{[z_0,z]} f(w)dw + \int_{[z,z+h]} f(w)dw + \int_{[z+h,z_0]} f(w)dw.$$

The segment $[z+h, z_0]$ can be obtained by an orientation reversed reparametrization of $[z_0, z+h]$, hence

$$-\int_{[z+h,z_0]} f(w)dw = \int_{[z_0,z+h]} f(w)dw$$

by Lemma 11.20 and (12.7) becomes

$$\int_{[z_0,z+h]} f(w)dw - \int_{[z_0,z]} f(w)dw = \int_{[z,z+h]} f(w)dw,$$

i.e.

$$F(z+h) - F(z) = \int_{[z,z+h]} f(w)dw.$$

To show that F is an antiderivative of f, consider

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_{[z,z+h]} f(w)dw - f(z).$$

It will be convenient to replace $f(z)$ by another integral along the same segment $[z, z+h]$. For this, take the affine parametrization

$$\gamma(t) = (1-t)z + t(z+h), \quad t \in [0,1],$$

and compute

$$\int_{[z,z+h]} dw = \int_0^1 \gamma'(t)dt = \int_0^1 h dt = h.$$

Then

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\left(\int_{[z,z+h]} f(w)dw - f(z)\int_{[z,z+h]} dw\right)$$
$$= \frac{1}{h}\int_{[z,z+h]} (f(w)-f(z))dw,$$

and by the basic estimate,

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \le \frac{1}{h}\sup_{[z,z+h]} |f(w)-f(z)|h = \sup_{[z,z+h]} |f(w)-f(z)|.$$

It suffices to observe that the continuity of f at z implies that the right-hand side converges to 0 as $h \to 0$, proving $F'(z) = f(z)$ as desired. □

Corollary 12.6 (Cauchy's theorem for star-shaped sets)**.** *Let $U \subset \mathbb{C}$ be open and star-shaped and f be* holomorphic *in U. Then for every* closed *piecewise C^1 path $\gamma\colon [a,b] \to U$, we have*

$$\int_\gamma f(z)dz = 0.$$

Proof. By the Cauchy-Goursat theorem,

$$\int_{\partial T} f(z)dz = 0$$

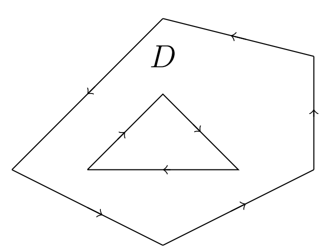
for any triangle $\overline{T} \subset U$. Then Lemma 12.5 implies that f has an anti-derivative F in U. Applying the fundamental theorem of calculus, Theorem 11.9, we compute

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)) = 0,$$

since γ is a closed path. □

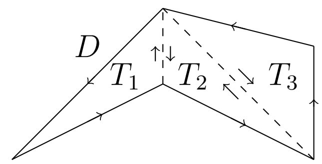
12.4. **Cauchy's theorem for polygonal sets.**

Definition 12.7. An open set D is *polygonal* if a union of piecewise affine paths represents its oriented boundary in the sense of Definition 12.1.



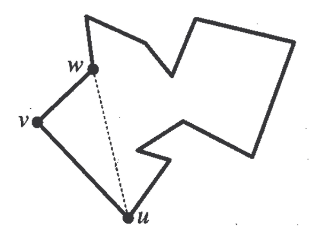
Definition 12.8. Let D be a polygonal set.

(1) A *diagonal* of D is and open line segment connecting two vertices of D that lies inside D.
(2) A *triangulation* of D is a decomposition of D into disjoint union of triangles by means of removing from D several diagonals.

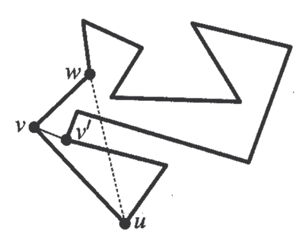


Lemma 12.9 (Existence of triangulations)**.** *Every polygonal set D admits a triangulation into a finite collection of triangles $T_1, \dots, T_n$, such that every edge of each T_j is either part of the oriented boundary ∂D or a diagonal which also occurs as edge of another T_k with reversed orientation.*

Proof. Remove from D a maximal number of diagonals. Then D splits into finitely many polygonal sets $T_1, \dots, T_n$ as in the lemma. Assume by contradiction that T_j is not a triangle for some j.



Let v be the leftmost vertex of T_j (take one if there are several), and u, w are two neighboring vertices of v. If the segment connecting u, w is a diagonal of T_j, it is another diagonal of T, a contradiction with the maximality of the number of diagonals.



Otherwise, there must exist vertices v' inside the triangle with vertices v, w, u. Let v' to be the farthest one from the line segment uw, we obtain another diagonal of T connecting v with v', again a contradiction.

□

Theorem 12.10 (Cauchy's theorem for polygonal sets)**.** *Let $U \subset \mathbb{C}$ be open and f be* holomorphic *in U. Then for every polygonal set D with $\overline{D} \subset U$, the integral of f over its oriented boundary vanishes:*

$$\int_{\partial D} f(z)dz = 0.$$

Proof. Choose triangles $T_1, \ldots, T_n$ as in Lemma 12.9. Taking the sum of integrals

$$\sum_j \int_{\partial T_j} f(z)dz,$$

observe that contributions along cuts are counted twice with opposite orientation, hence cancel each other. What remains are the boundary edges of D, whose contributions' sum is $\int_{\partial D} f(z)dz$. Therefore

$$\int_{\partial D} f(z)dz = \sum_j \int_{\partial T_j} f(z)dz.$$

Now each integral in the sum vanishes by the Cauchy-Goursat theorem, proving the conclusion. □

12.5. **Winding numbers.** We would like to generalize Cauchy's theorem for general closed paths γ. If $\gamma = \partial D$ for a polygonal set D, we require that the entire set $\overline{D}$ is contained in the domain of definition of the given holomorphic function. If γ is more general, we need a replacement for the set $\overline{D}$.

For this, we introduce the *winding number* and show that the winding number of ∂D around a point z_0 is nonzero if and only if $z_0 \in D$.

Definition 12.11 (winding number)**.** Let $\gamma\colon [a, b] \to \mathbb{C}$ be a closed piecewise C^1 path and $z_0 \notin \gamma([a, b])$. Define the *winding number* of γ around z_0 by

$$W_\gamma(z_0) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

Intuitively, $W_\gamma(z_0)$ equals the total number of times γ travels counterclockwise around z_0. To see this, we shall compute winding numbers by means of local antiderivatives of $\frac{1}{z-z_0}$ as branches of $\log(z - z_0)$.

Lemma 12.12. *Let $\gamma\colon [a, b] \to \mathbb{C}$ be a path and $z_0 \notin \gamma([a, b])$. Then there exists a partition*

$$a = t_0 < t_1 < \ldots < t_n = b \tag{12.8}$$

and open disks $\Delta_1, \ldots, \Delta_n$ such that for all $j = 1, \ldots n$:

(1) $\gamma([t_{j-1}, t_j]) \subset \Delta_j$;
(2) $z_0 \notin \Delta_j$.

Proof. We apply Lemma 5.14 on the existence of a uniform radius of disks to the compact set $\gamma([a,b])$ contained in the open set $U := \mathbb{C} \setminus \{z_0\}$. By the Lemma, there exists $r > 0$ such that

$$z \in \gamma([a,b]) \implies \Delta_r(z) \subset U. \tag{12.9}$$

Next, since $[a,b]$ is compact and γ is continuous, it is also *uniformly continuous* by Theorem 8.25. Hence, there exists $\delta > 0$ such that

$$t, s \in [a,b],\ |t-s| < \delta \implies |\gamma(t) - \gamma(s)| < r. \tag{12.10}$$

Now choose any partition (12.8) with $|t_{j-1} - t_j| < \delta$ for all j. Then (12.10) implies

$$\gamma([t_{j-1}, t_j]) \subset \Delta_r(\gamma(t_j)) =: \Delta_j,$$

and $\Delta_j \subset U$ by (12.9) implies $z_0 \notin \Delta_j$, as desired. □

Lemma 12.13. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise C^1 path, and z_0, t_j and Δ_j satisfy the conclusion of Lemma 12.12. Then for each j, there exists a brach of* $\log(z - z_0)$,

$$F_j\colon \Delta_j \to \mathbb{C}$$

such that

(1) *F_j is holomorphic in Δ_j with $F_j'(z) = \frac{1}{z-z_0}$;*
(2) *$\int_{\gamma_j} \frac{dz}{z-z_0} = F_j(\gamma_j(t_j)) - F_j(\gamma_j(t_{j-1}))$, where $\gamma_j := \gamma_{[t_{j-1}, t_j]}$.*

Proof. Assume $z_0 = 0$ and $\Delta_j = \Delta_r(x)$ for some real $x > 0$. Then $\Delta_j \subset \mathbb{C} \setminus (-\infty, 0]$, where we can choose $F_j(z) := \mathsf{Log} z$. Then we have proved that F_j is holomorphic with $F_j'(z) = \frac{1}{z}$ and (2) holds by the complex fundamental theorem of calculus Theorem 11.9.

For a general disk $\Delta_j = \Delta_r(z_j)$ with $0 \notin \Delta_j$, choose a branch $\varphi_j\colon \Delta_j \to \mathbb{R}$ of $\arg z$, e.g. the one determined by

$$\mathsf{Arg} z_j - \pi < \varphi_j(z) < \mathsf{Arg} z_j + \pi.$$

Then

$$F_j(z) := \ln|z| + i\varphi_j(z)$$

defines a branch of $\log z$ satisfying the conclusion of the lemma.

Finally the case of general z_0 is obtained from $z_0 = 0$ by translation:

$$F_j(z) := \widetilde{F}_j(z - z_0), \quad z \in \Delta_j = \Delta_r(z_j), \quad z_0 \notin \Delta_j,$$

where $\widetilde{F}_j$ is a branch of $\log z$ on $\Delta_r(z_j - z_0)$ with $0 \notin \Delta_r(z_j - z_0)$. □

Corollary 12.14. *For every closed piecewise C^1 path $\gamma\colon [a,b] \to \mathbb{C}$ and $z_0 \notin \gamma([a,b])$, the winding number $W_\gamma(z_0)$ is an integer.*

Proof. Choosing the data as in Lemma 12.13, we compute

$$\int_\gamma \frac{dz}{z-z_0} = \sum_{j=1}^{n} \int_{\gamma_j} \frac{dz}{z-z_0} = \sum_{j=1}^{n} (F_j(\gamma(t_j)) - F_j(\gamma(t_{j-1})))$$

$$= (F_1(\gamma(t_1)) - F_1(\gamma(t_0))) + (F_2(\gamma(t_2)) - F_2(\gamma(t_1))) + \ldots + (F_n(\gamma(t_n)) - F_n(\gamma(t_{n-1})))$$

Using $\gamma(a) = \gamma(b)$ since γ is closed and setting $F_{n+1} := F_1$ we can regroup the sum in the right-hand side as

$$(F_1(\gamma(t_1)) - F_2(\gamma(t_1))) + \ldots + (F_{n-1}(\gamma(t_{n-1})) - F_n(\gamma(t_{n-1}))) + (F_n(\gamma(t_n)) - F_1(\gamma(t_0)))$$

$$= \sum_{j=1}^{n} (F_j(\gamma(t_j)) - F_{j+1}(\gamma(t_j)).)$$

Here each summand is a difference between values at $\gamma(t_j))$ of different branches of $\log(z-z_0)$. Since $\log(z-z_0)$ consists of values up to an integer multiple of $2\pi i$, there exist integers n_j such that

$$F_j(\gamma(t_j)) - F_{j+1}(\gamma(t_j)) = 2\pi i n_j, \quad n_j \in \mathbb{Z},$$

proving that $W_\gamma(z_0)$ is an integer as desired. □

Using the construction of branches of logarithm on disks as in the proof of Lemma 12.13, we obtain:

Lemma 12.15. *Let Δ be a disk and γ a closed path in Δ. Then*

$$W_\gamma(z_0) = 0$$

for all $z_0 \notin \Delta$.

Next we investigate the dependence of $W_\gamma(z_0)$ on z_0.

Lemma 12.16. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed piecewise C^1 path. The winding number $W_\gamma(z_0)$ depends continuously on $z_0 \in \mathbb{C} \setminus \gamma([a,b])$.*

Proof. Fix

$$z_0 \in U := \mathbb{C} \setminus \gamma([a,b]).$$

Since $\gamma([a,b])$ is compact and hence closed, U is open. Then there exists $c > 0$ with $\Delta_c(z_0) \subset U$. Taking z_1 with $|z_1 - z_0| < c/2$, we have

$$\left| \frac{1}{z-z_1} - \frac{1}{z-z_0} \right| = \left| \frac{z_1 - z_0}{(z-z_1)(z-z_0)} \right| \le \frac{4|z_1-z_0|}{c^2},$$

and

$$|W_\gamma(z_1) - W_\gamma(z_0)| = \frac{1}{2\pi} \left| \int_\gamma \left(\frac{1}{z-z_1} - \frac{1}{z-z_0} \right) dz \right|$$

$$\le \frac{4|z_1-z_0|}{2\pi c^2} L(\gamma)$$

by the basic estimate Theorem 11.23, and the continuity follows as claimed. □

Corollary 12.17. *Let* $\gamma\colon [a,b] \to \mathbb{C}$ *be piecewise* C^1 *and* $S \subset \mathbb{C}\setminus\gamma([a,b])$ *is connected. Then the winding number* $W_\gamma(z_0)$ *is* constant *for* $z_0 \in S$.

Proof. Indeed, $f(z_0) := W_\gamma(z_0)$ is continuous on S and takes values in $\mathbb{Z}$. Hence $f(S) \subset \mathbb{Z}$ is connected, which is only possible when $f(S)$ is a single point, i.e. $f = \text{const}$ on S. □

12.6. **Winding numbers for oriented boundaries.**

Lemma 12.18. *Let* T *be either*

1. *an open triangle or*
2. *an open disk,*

and let γ *be the oriented boundary of* T. *Then*

$$W_\gamma(z_0) = \begin{cases} 1 & z_0 \in T \\ 0 & z_0 \in \mathbb{C}\setminus\overline{T} \end{cases}.$$

Proof. Assume first T is a triangle. If $z_0 \in \mathbb{C}\setminus\overline{T}$, we have $\overline{T} \subset U := \mathbb{C}\setminus\{z_0\}$ and the function

$$f(z) = \frac{1}{z - z_0}$$

is holomorphic in U, hence

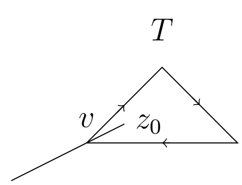
$$W_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma f(z)dz = 0$$

by Cauchy-Goursat.

On the other hand, when $z_0 \in T$, there exists a branch $F(z)$ of

$$\log(z - z_0), \quad z \in \mathbb{C}\setminus R,$$

where R is a ray starting at z_0 and passing through the vertex v of T.



If $\gamma\colon [a,b] \to \mathbb{C}$ represents the oriented boundary of T with $\gamma(a) = \gamma(b) = v$, then for $\delta > 0$ sufficiently small,

$$\gamma_\delta := \gamma|_{[a+\delta,b-\delta]}, \quad \gamma_\delta\colon [a+\delta, b-\delta] \to \mathbb{C}\setminus R,$$

and

$$\int_{\gamma_\delta} \frac{dz}{z - z_0} = F(\gamma(b-\delta)) - F(\gamma(a+\delta)).$$

Taking the limit as $\delta \to 0$ and using the fact that the limits of the branch F differ by $2\pi i$, we obtain the conclusion for $z_0 \in T$.

Now assume $T = \Delta_r(z_1)$ is a disk. If $z_0 \in T$, the proof of $W_\gamma(z_0) = 1$ is analogous to the above case when T is a triangle.

The conclusion for points z_0 in the complement of $\overline{T}$ follows from Lemma 12.15. □

Theorem 12.19. *Let D be a polygonal set with oriented boundary represented by the disjoint union of closed paths $\gamma_1, \ldots, \gamma_n$. Then*

$$\sum_k W_{\gamma_k}(z_0) = \begin{cases} 1 & z_0 \in D \\ 0 & z_0 \in \mathbb{C} \setminus \overline{D} \end{cases}.$$

Proof. By Lemma 12.9, D admits a triangulation with triangles $T_1, \ldots, T_n$.

If $z_0 \in \mathbb{C} \setminus \overline{D}$, the function $\frac{1}{z-z_0}$ is holomorphic in the open set $\mathbb{C} \setminus \{z_0\}$ containing $\overline{D}$ and the winding number is zero by Cauchy's theorem.

Assume next $z_0 \in D$ does not belong to any edge of any T_j. Then

$$\sum_k W_{\gamma_k}(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{dz}{z - z_0} = \sum_j \frac{1}{2\pi i} \int_{\partial T_j} \frac{dz}{z - z_0}.$$

Since z_0 belongs to precisely one triangle T_j, this sum equals 1 in view of Lemma 12.18.

Finally if $z_0 \in D$ belongs to an edge of some T_j and choose any $z_1 \in T_j$. Then we have already shown that $\sum_k W_{\gamma_k}(z_1) = 1$. Since the line segment S connecting z_0 with z_1 is connected and $S \subset \mathbb{C} \setminus \partial D$, Corollary 12.17 implies that

$$\sum_k W_{\gamma_k}(z_0) = \sum_k W_{\gamma_k}(z_1) = 1$$

as desired. □

12.7. **Cauchy's Integral Formula for polygonal sets.** While it can be difficult to compute specific integrals, it is often easier to compute their limits when the paths "converge to a point":

Lemma 12.20. *Let $U \subset \mathbb{C}$ be open with $z_0 \in U$, $f\colon U \to \mathbb{C}$ be continuous function, and $\gamma\colon [a,b] \to \mathbb{C}$ a piecewise C^1 path such that*

1. $f(z_0) = 0$;
2. $0 \notin \gamma([a,b])$.

Define

$$\gamma_\varepsilon(t) := z_0 + \varepsilon\gamma(t).$$

Then

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon}\frac{f(z)}{z-z_0}dz = 0.$$

Proof. Since $0 \notin \gamma([a,b])$, there exists $c > 0$ with $|\gamma(t)| \geq c$ for all $t \in [a,b]$, hence by scaling,

$$\sup_{z\in\gamma_\varepsilon([a,b])}\left|\frac{1}{z-z_0}\right| \leq \frac{1}{\varepsilon c}.$$

Also the length of γ_ε can be estimated by

$$L(\gamma_\varepsilon) \leq \varepsilon C$$

for some $C > 0$. Then using the basic estimate Lemma 11.23 for the integral,

$$\left|\int_{\gamma_\varepsilon}\frac{f(z)}{z-z_0}dz\right| \leq \sup_{z\in\gamma_\varepsilon([a,b])}\left|\frac{f(z)}{z-z_0}\right| L(\gamma_\varepsilon) \leq \frac{C}{c}\sup_{z\in\gamma_\varepsilon([a,b])}|f(z)|,$$

which converges to 0 since $\lim_{z\to z_0} f(z) = 0$. □

Theorem 12.21 (Cauchy's Integral Formula for polygonal sets)**.** *Let $U \subset \mathbb{C}$ be open, $f\colon U \to \mathbb{C}$ holomorphic and P an open polygonal set with $\overline{P} \subset U$. Then*

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial P}\frac{f(z)dz}{z-z_0}$$

for any $z_0 \in P$.

Note that the denominator $z - z_0$ never vanishes for z_0 in the open set P and z in the boundary ∂P.

Proof. Let T be an open triangle with $0 \in T$ and γ be the oriented boundary of T. Fix $z_0 \in P$. Then by Lemma 12.20,

$$\lim_{\varepsilon\to 0}\int_{\partial T_\varepsilon}\frac{f(z)-f(z_0)}{z-z_0}dz = 0, \quad T_\varepsilon := z_0 + \varepsilon T.$$

Here T_ε is a triangle containing z_0. Next,

$$\int_{\partial T_\varepsilon}\frac{f(z_0)}{z-z_0}dz = f(z_0)\int_{\partial T_\varepsilon}\frac{1}{z-z_0}dz = f(z_0)\cdot 2\pi i W_{\partial T_\varepsilon}(z_0) = 2\pi i f(z_0)$$

by Lemma 12.18, implying

$$\lim_{\varepsilon\to 0}\int_{\partial T_\varepsilon}\frac{f(z)}{z-z_0}dz = 2\pi i f(z_0) \tag{12.11}$$

Furthermore, for $\varepsilon > 0$ sufficiently small,

$$P' := P \setminus \overline{T}_\varepsilon \subset U \setminus \{z_0\}$$

is a polygonal set and the function

$$\frac{f(z)}{z - z_0}$$

is holomorphic in $U \setminus \{z_0\}$. Then the oriented boundary of P' consists of the oriented boundary of P and the oriented boundary of T_ε with reverse orientation. Applying Cauchy's Theorem for polygonal sets (Theorem 12.10), we conclude that

$$\left(\int_{\partial P} - \int_{\partial T_\varepsilon} \right) \frac{f(z)dz}{z - z_0} = 0.$$

Using (12.11) and taking limits as $\varepsilon \to 0$, we obtain

$$\int_{\partial P} \frac{f(z)dz}{z - z_0} - 2\pi i f(z_0) = 0$$

as desired. □

12.8. **Cauchy's theorem for arbitrary paths.**

Definition 12.22. Let

$$\gamma_j \colon [a_j, b_j] \to \mathbb{C}, \quad j = 1, \ldots, n,$$

be closed piecewise C^1 paths in $\mathbb{C}$. Define their interior by

$$\mathsf{Int}(\gamma_1, \ldots, \gamma_n) := \{z_0 \in \mathbb{C} \setminus \cup_j \gamma_j([a_j, b_j]) : \sum_j W_{\gamma_j}(z_0) \neq 0\}.$$

Theorem 12.23 (Cauchy's Theorem: general case)**.** *Let $U \subset \mathbb{C}$ be open, $f \colon U \to \mathbb{C}$ holomorphic, and $\gamma_1, \ldots, \gamma_n$ closed piecewise C^1 paths in U such that*

$$\mathsf{Int}(\gamma_1, \ldots, \gamma_n) \subset U. \tag{12.12}$$

Then

$$\sum_j \int_{\gamma_j} f(z)dz = 0.$$

The main condition (12.12) is clearly equivalent to

$$z_0 \notin U \implies \sum_j W_{\gamma_j}(z_0) = 0. \tag{12.13}$$

We shall need the following lemma:

Lemma 12.24. *Let $U \subset \mathbb{C}$ be open and $K \subset U$ be compact. Then there exists an open polygonal set P with*

$$K \subset P \subset \overline{P} \subset U.$$

Proof. By Lemma 5.14, there exists $r > 0$ such that

$$z \in K \implies \Delta_r(z) \subset U. \tag{12.14}$$

Set $\delta := r/2$ and consider the squares

$$S_{kl} := \{z \in \mathbb{C} : k\delta \leq \mathsf{Re}\, z \leq (k+1)\delta,\ l\delta \leq \mathsf{Im}\, z \leq (l+1)\delta\}, \quad k, l \in \mathbb{Z}.$$

Since K is bounded, there exists finitely many S_{kl} intersecting K. Let $\widetilde{P}$ be the union of all such squares S_{kl}, and P be the interior of $\widetilde{P}$. Then P satisfies the desired conclusion. □

Proof of Theorem 12.23. We would like to apply Lemma 12.24 to the set

$$K := \mathsf{Int}(\gamma_1, \ldots, \gamma_n) \bigcup \cup_j \gamma_j([a_j, b_j]).$$

For this, observe that the complement

$$\mathbb{C} \setminus K = \{z_0 \in \mathbb{C} \setminus \bigcup_j \gamma_j([a_j, b_j]) : \sum_j W_{\gamma_j}(z_0) = 0\}.$$

is open as consequence of Corollary 12.17, since each point in $V := \mathbb{C} \setminus \bigcup_j \gamma_j([a_j, b_j])$ is surrounded by a connected open disk in V, where all winding numbers are constant.

Furthermore, since $\cup_j \gamma_j([a_j, b_j])$ is compact, it is contained in a disk Δ and all winding numbers $W_{\gamma_j}(z_0) = 0$ for $z_0 \notin D$ by Lemma 12.15, implying $K \subset \Delta$. Then K is closed and bounded, hence it is compact by Heine-Borel's theorem.

Now we can apply Lemma 12.24 to obtain an open polygonal set P with

$$K \subset P \subset \overline{P} \subset U.$$

Next, by Cauchy's integral formula Theorem 12.21, for every $z \in K$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial P} \frac{f(w)dw}{w - z}.$$

Since $\gamma_j([a_j, b_j]) \subset K$ for all j,

$$\sum_j \int_{\gamma_j} f(z)dz = \sum_j \int_{\gamma_j} \left(\frac{1}{2\pi i} \int_{\partial P} \frac{f(w)dw}{w - z} \right) dz$$

$$= \frac{1}{2\pi i} \sum_j \int_{\partial P} \left(\int_{\gamma_j} \frac{f(w)dz}{w - z} \right) dw = \frac{1}{2\pi i} \sum_j \int_{\partial P} f(w) \left(\int_{\gamma_j} \frac{dz}{w - z} \right) dw.$$

$$= -\int_{\partial P} f(w) \left(\sum_j W_{\gamma_j}(w) \right) dw,$$

where we used the standard result on switching the order of integration for continuous functions.

Finally, since $\partial P \cap K = \emptyset$, for $w \in \partial P$, we must have $w \notin K$, hence

$$\sum_j W_{\gamma_j}(w) = 0$$

and the above integral vanishes proving the desired conclusion. □

Example 12.25. Let Δ_1, Δ_2 be open disks in $\mathbb{C}$,

$$z_0 \in \Delta_1 \cap \Delta_2,$$

and consider the holomorphic function

$$f(z) = \frac{z \sin z}{z - z_0}, \quad z \in U := \mathbb{C} \setminus \{z_0\}.$$

Denote by γ_1 the oriented boundary of Δ_1 and by γ_2 the boundary of Δ_2 with reverse orientation. Then by Lemma 12.18,

$$W_{\gamma_1}(z_0) = 1, \quad W_{\gamma_2}(z_0) = -1 \implies W_{\gamma_1}(z_0) + W_{\gamma_2}(z_0) = 0$$
$$\implies \mathsf{Int}(\gamma_1, \gamma_2) = \{z \in \mathbb{C} : W_{\gamma_1}(z) + W_{\gamma_2}(z) \neq 0\} \subset \mathbb{C} \setminus \{z_0\} = U.$$

Hence all assumptions of Cauchy's Theorem are satisfied and we conclude

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = 0.$$

13. Cauchy's Residue Theorem

13.1. **Residues.** In Cauchy's theorem, the function is assumed holomorphic in the given open set. However, in many applications, functions are only holomoprhic outside a finite sets of points. A simple example is the ration function $f(z) = \frac{P(z)}{Q(z)}$ which is holomorphic outside the zero set of Q.

Definition 13.1. A point z_0 is called an *isolated singularity* for a function f if f is holomorphic in a punctured disk $\Delta_r(z_0) \setminus \{z_0\}$ for some $r > 0$.

Definition 13.2 (Residue)**.** Let f be holomorphic in a punctured disk $\Delta_r(z_0) \setminus \{z_0\}$. The Residue of f at z_0 is given

$$\mathsf{Res}_{z_0} f := \frac{1}{2\pi i} \int_{\partial \Delta_\varepsilon(z_0)} f(z)dz,$$

where $0 < \varepsilon < r$.

Note that $\mathsf{Res}_{z_0} f$ is well-defined, i.e. is independent of ε. Indeed, if $0 < \varepsilon_1 < \varepsilon_2 < r$, let γ_2 be the oriented boundary of $\Delta_{\varepsilon_2}(z_0)$ and γ_1 the oriented boundary of $\Delta_{\varepsilon_1}(z_0)$ with reverse orientation. Then

$$\mathsf{Int}(\gamma_1, \gamma_2) = \Delta_{\varepsilon_2}(z_0) \setminus \overline{\Delta_{\varepsilon_1}(z_0)} \subset \Delta_r(z_0) \setminus \{z_0\},$$

and by Cauchy's theorem,

$$\int_{\partial\Delta_{\varepsilon_2}(z_0)} f(z)dz - \int_{\partial\Delta_{\varepsilon_1}(z_0)} f(z)dz = 0,$$

hence the integral in Definition 13.2 is indeed independent of ε.

As warming-up example, consider the simplest situation:

Lemma 13.3. *Let the function f be holomorphic in the disk $\Delta_r(z_0)$. Then*

$$\mathsf{Res}_{z_0} f = 0.$$

Proof. Indeed, $\Delta_r(z_0)$ is star-shaped and by the Cauchy's Theorem,

$$\mathsf{Res}_{z_0} f = \frac{1}{2\pi i}\int_{\partial\Delta_\varepsilon(z_0)} f(z)dz = 0,$$

hence the residue of f is always 0 at every point of an open set where f is holomorphic. □

In view of this lemma, only points where function is not holomorphic (or not defined) are those where the residue may not be zero.

Example 13.4. Using our computation of the winding number in Lemma 12.18, we obtain

$$\mathsf{Res}_{z_0} \frac{1}{z - z_0} = \frac{1}{2\pi i}\int_{\partial\Delta_\varepsilon(z_0)} \frac{1}{z - z_0} dz = W_{\partial\Delta_\varepsilon(z_0)}(z_0) = 1,$$

which explains the division by the constant $2\pi i$ in Definition 13.2.

We have the following generalisation of the above example:

Lemma 13.5. *Let f be holomorphic in the punctured disk $\Delta_r(z_0) \setminus \{z_0\}$ and continuous in $\Delta_r(z_0)$. Then:*

(1)
$$\mathsf{Res}_{z_0} \frac{f(z)}{z - z_0} = f(z_0);$$

(2) *if g is holomorphic in $\Delta_r(z_0)$ with $g(z_0) = 0$, $g'(z_0) \neq 0$, and $g(z) \neq 0$ for $z \in \Delta_r(z_0) \setminus \{z_0\}$,*
$$\mathsf{Res}_{z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}$$

Proof. For (1), compute

$$\mathsf{Res}_{z_0}\frac{f(z)}{z-z_0} = \frac{1}{2\pi i}\int_{\partial\Delta_\varepsilon(z_0)}\frac{f(z)}{z-z_0}dz$$
$$= \frac{1}{2\pi i}\left(\int_{\partial\Delta_\varepsilon(z_0)}\frac{f(z_0)}{z-z_0}dz + \int_{\partial\Delta_\varepsilon(z_0)}\frac{f(z)-f(z_0)}{z-z_0}dz\right),$$

where $\varepsilon > 0$ is sufficiently small. By the above example, the first integral is

$$2\pi i f(z_0)$$

independently of ε. Then the second integral is also independent of ε, and hence it is equal to its limit as $\varepsilon \to 0$. Now Lemma 12.20 implies that the second integral must vanish, proving the conclusion in (1).

To show (2), since $g(z_0)$, we can write

$$\frac{f(z)}{g(z)} = \frac{h(z)}{z-z_0}, \quad h(z) := f(z)\frac{z-z_0}{g(z)-g(z_0)}.$$

Here $h(z)$ is defined for $z \neq z_0$ and we can extend it to a continuous function in $\Delta_r(z_0)$ by setting $h(z_0) := \frac{f(z_0)}{g'(z_0)}$. Then the extension of h satisfies the assumptions of (1) and (2) follows from (1) applied to this case. □

Example 13.6. For every $z_0 \in \mathbb{C}$ and $n \in \mathbb{Z}$, $n \geq 2$, the function

$$f(z) = \frac{1}{(z-z_0)^n} = (z-z_0)^{-n}$$

has the antiderivative

$$F(z) = \frac{1}{(-n+1)(z-z_0)^{-n+1}}, \quad z \in \mathbb{C}\setminus\{z_0\}.$$

Hence the integral of f vanishes along any closed path, in particular,

$$\mathsf{Res}_{z_0}\frac{1}{(z-z_0)^n} = \frac{1}{2\pi i}\int_{\partial\Delta_\varepsilon(z_0)}\frac{1}{(z-z_0)^n}dz = 0.$$

13.2. **Cauchy's Residue Theorem.**

Theorem 13.7 (Cauchy's Residue Theorem)**.** *Let $U \subset \mathbb{C}$ be open, $z_1, \ldots, z_m \in U$,*

$$f\colon U\setminus\{z_1,\ldots,z_m\} \to \mathbb{C}$$

be holomorphic, and $\gamma_1, \ldots, \gamma_n$ be closed piecewise C^1 paths in $U \setminus \{z_1, \ldots, z_m\}$ such that

$$\mathsf{Int}(\gamma_1,\ldots,\gamma_n) \subset U. \tag{13.1}$$

Then

$$\sum_j \int_{\gamma_j} f(z)dz = 2\pi i \sum_{j,k} W_{\gamma_j}(z_k)\mathsf{Res}_{z_k} f.$$

Proof. The main idea is to add to γ_j small circles around z_k and apply Cauchy's theorem in $U \setminus \{z_1, \ldots, z_m\}$. For each z_k, $k = 1, \ldots, m$, consider a disk $\Delta_\varepsilon(z_k)$. Choose $\varepsilon > 0$ sufficiently small so that $\overline{\Delta_\varepsilon(z_k)} \subset U$ and

$$\overline{\Delta_\varepsilon(z_k)} \cap \overline{\Delta_\varepsilon(z_l)} = \emptyset, \quad k \neq l. \tag{13.2}$$

In order to apply Cauchy's theorem in $U \setminus \{z_1, \ldots, z_m\}$, we need to construct closed paths φ_l so that

$$\mathsf{Int}(\gamma_1, \ldots, \gamma_n, \varphi_1, \ldots, \varphi_s) \subset U \setminus \{z_1, \ldots, z_m\}.$$

For every $k = 1, \ldots, m$, set

$$l_k := \sum_j W_{\gamma_j}(z_k),$$

and let φ_{ks} for $s = 1, \ldots, |l_k|$, be either $|l_k|$ copies of the oriented boundary of $\Delta_\varepsilon(z_k)$ if $l_k < 0$ or $|l_k|$ copies of the oriented boundary of $\Delta_\varepsilon(z_k)$ with reverse orientation if $l_k > 0$. Then in each case

$$\sum_j W_{\gamma_j}(z_k) + \sum_s W_{\varphi_{ks}}(z_k) = 0.$$

Collecting such φ_{ks} for all k and using (13.2), we obtain

$$\sum_j W_{\gamma_j}(z_k) + \sum_{k,s} W_{\varphi_{ks}}(z_k) = 0$$

for all k. On the other hand, if $z_0 \notin U$, (13.1) implies

$$\sum_j W_{\gamma_j}(z_0) + \sum_{k,s} W_{\varphi_{ks}}(z_0) = 0.$$

Hence

$$\mathsf{Int}(\gamma_1, \ldots, \gamma_n, \varphi_1, \ldots, \varphi_s) \subset U \setminus \{z_1, \ldots, z_m\},$$

where $\varphi_1, \ldots, \varphi_s$ are all φ_{ks} reindexed.

Applying Cauchy's theorem to f in $U \setminus \{z_1, \ldots, z_m\}$ we conclude

$$\sum_j \int_{\gamma_j} f(z)dz + \sum_l \int_{\varphi_l} f(z)dz = 0, \tag{13.3}$$

and by our choice of φ_l,

$$\sum_l \int_{\varphi_l} f(z)dz = -2\pi i \sum_k l_k \mathsf{Res}_{z_k} f = -2\pi i \sum_{j,k} W_{\gamma_j}(z_k)\mathsf{Res}_{z_k} f,$$

which together with (13.3) proves the desired conclusion. □

Example 13.8. The function
$$f(z) = \frac{2e^z}{z^2-1} = \frac{2e^z}{(z-1)(z+1)} = \frac{e^z}{z-1} - \frac{e^z}{z+1}$$
is holomorphic in
$$\mathbb{C} \setminus \{-1, 1\}.$$
By Lemmata 13.3 and 13.5, we compute the residues
$$\mathsf{Res}_{-1} f = \mathsf{Res}_{-1} \frac{e^z}{z-1} - \mathsf{Res}_{-1} \frac{e^z}{z+1} = 0 - e^{-1} = -\frac{1}{e},$$
$$\mathsf{Res}_1 f = \mathsf{Res}_1 \frac{e^z}{z-1} - \mathsf{Res}_1 \frac{e^z}{z+1} = e^1 - 0 = e.$$
Now we use the Residue Theorem for $U = \mathbb{C}$ to compute integrals of f along boundaries of disks.

First, consider the disk $\Delta_3(0)$ with center 0 and radius 3, which contains both points -1 and 1. If γ_1 is the oriented boundary of $\Delta_3(0)$, we have shown in Lemma 12.18 that
$$W_{\gamma_1}(-1) = W_{\gamma_1}(1) = 1.$$
Then by Cauchy's Residue Theorem,
$$\int_{\partial\Delta_3(0)} f(z)dz = \int_{\gamma_1} f(z)dz = 2\pi i(W_{\gamma_1}(-1)\mathsf{Res}_{-1} f + W_{\gamma_1}(1)\mathsf{Res}_1 f)$$
$$= 2\pi i(\mathsf{Res}_{-1} f + \mathsf{Res}_1 f) = 2\pi i(e - 1/e).$$
Next, consider the disk $\Delta_1(-1)$ with center -1 and radius 1, which contains only -1 but not 1. If γ_2 is the oriented boundary of $\Delta_1(-1)$, by Lemma 12.18,
$$W_{\gamma_1}(-1) = 1, \quad W_{\gamma_1}(1) = 0,$$
hence by Cauchy's Residue Theorem,
$$\int_{\partial\Delta_1(-1)} f(z)dz = \int_{\gamma_2} f(z)dz = 2\pi i(W_{\gamma_2}(-1)\mathsf{Res}_{-1} f + W_{\gamma_2}(1)\mathsf{Res}_1 f)$$
$$= 2\pi i \mathsf{Res}_{-1} f = -\frac{2\pi i}{e}.$$
Finally, consider the disk $\Delta_1(1)$ with center 1 and radius 1, which contains only 1 but not -1. If γ_3 is the oriented boundary of $\Delta_1(1)$, by Lemma 12.18,
$$W_{\gamma_3}(-1) = 0, \quad W_{\gamma_3}(1) = 1,$$

hence by Cauchy's Residue Theorem,

$$\int_{\partial\Delta_1(1)} f(z)dz = \int_{\gamma_3} f(z)dz = 2\pi i(W_{\gamma_3}(-1)\mathsf{Res}_{-1}f + W_{\gamma_3}(1)\mathsf{Res}_1 f)$$
$$= 2\pi i \mathsf{Res}_1 f = 2\pi i e.$$

14. Application of Cauchy's Residue Theorem to integrals

The Residue Theorem is a powerful tool that can be applied to evaluate real integrals of real functions.

14.1. **Trigonometric integrals.** Consider integrals of the form

$$\int_0^{2\pi} R(\cos t, \sin t)dt, \tag{14.1}$$

where R is a rational function.

To evaluate (14.1), rewrite (14.1) as the complex integral along the path

$$\gamma(t) = \cos t + i\sin t = e^{it}, \quad t \in [0, 2\pi],$$

and use the substitution

$$z = e^{it}, \quad \cos t = \frac{z+z^{-1}}{2}, \quad \sin t = \frac{z-z^{-1}}{2i} : \tag{14.2}$$

$$\int_0^{2\pi} R(\cos t, \sin t)dt = \int_0^{2\pi} R(\cos t, \sin t)\frac{1}{\gamma'(t)}\gamma'(t)dt$$
$$= \int_{\partial\Delta_1(0)} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{1}{iz}dz,$$

where the latter integral can be evaluated by means of the Residue Theorem.

Example 14.1. Consider the integral

$$I = \int_0^{2\pi} \frac{dt}{1 - 2p\cos t + p^2},$$

where p is a real or complex parameter. Substituting (14.2) as above, we obtain

$$I = \int_{\partial D_1(0)} \frac{1}{1 - 2p\frac{z+z^{-1}}{2} + p^2}\frac{dz}{iz} = \frac{1}{i}\int_{\partial D_1(0)} \frac{dz}{z - pz^2 - p + p^2 z}$$
$$= \frac{1}{i}\int_{\partial D_1(0)} \frac{dz}{(z-p)(1-pz)} = \frac{1}{i}\int_{\partial D_1(0)} f(z)dz,$$

where

$$f(z) = \frac{1}{(z-p)(1-pz)}$$

is holomorphic in $\mathbb{C} \setminus \{p, p^{-1}\}$ if $p \neq 0$. (If $p = 0$, the integral is easy to calculate: $I = 2\pi$.)

Now with γ being the oriented boundary of the unit disk $\Delta_1(0)$, we compute the winding numbers

$$W_\gamma(p) = \begin{cases} 1 & p \in \Delta_1(0) \iff |p| < 1 \\ 0 & p \notin \overline{\Delta_1(0)} \iff |p| > 1 \end{cases},$$

and

$$W_\gamma(p^{-1}) = \begin{cases} 1 & p^{-1} \in \Delta_1(0) \iff |p| > 1 \\ 0 & p^{-1} \notin \overline{\Delta_1(0)} \iff |p| < 1 \end{cases}.$$

When $|p| = 1$, the function f is not continuous on the unit circle, the integral is improper and we cannot apply our method directly. Otherwise, by the Residue Theorem,

$$I = \frac{1}{i} 2\pi i \left(W_\gamma(p) \mathsf{Res}_p f + W_\gamma(p^{-1}) \mathsf{Res}_{p^{-1}} f \right). \tag{14.3}$$

By Lemma 13.5, calculate the residues

$$\mathsf{Res}_p \frac{1}{(z-p)(1-pz)} = \mathsf{Res}_p \frac{g(z)}{(z-p)} = g(p) = \frac{1}{1-p^2},$$

where

$$g(z) = \frac{1}{1-pz},$$

and

$$\mathsf{Res}_{p^{-1}} \frac{1}{(z-p)(1-pz)} = \mathsf{Res}_{p^{-1}} \frac{k(z)}{h(z)} = \frac{k(p^{-1})}{h'(p^{-1})} = \frac{1}{p^{-1}-p} \cdot \frac{1}{-p} = \frac{1}{p^2-1},$$

where

$$k(z) = \frac{1}{z-p}, \quad h(z) = 1 - pz.$$

Then finally, (14.3) becomes

$$I = \begin{cases} \frac{2\pi}{1-p^2} & |p| < 1 \\ \frac{2\pi}{p^2-1} & |p| > 1 \end{cases}.$$

14.2. **Improper integrals.**

Definition 14.2. The principal value of the improper integral is defined by

$$p.v. \int_{-\infty}^{\infty} f(x)dx := \lim_{c\to+\infty} \int_{-c}^{c} f(x)dx.$$

Lemma 14.3. *Let U be an open set in $\mathbb{C}$,*

$$z_1, \ldots, z_m \in \{z : \mathsf{Im}\, z > 0\} \subset \{z : \mathsf{Im}\, z \geq 0\} \subset U,$$

and F a holomorphic function in $U \setminus \{z_1, \ldots, z_m\}$. Assume

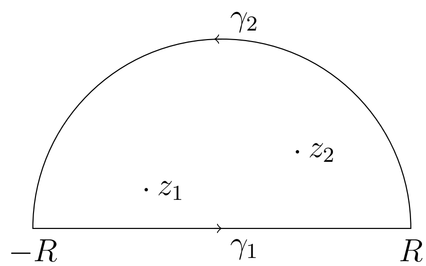
$$\lim_{z\to\infty} zF(z) = 0. \tag{14.4}$$

Then

$$p.v. \int_{-\infty}^{\infty} F(x)dx = 2\pi i \sum_k \mathsf{Res}_{z_k} F.$$

Proof. Consider the half-disk

$$D := \{z \in \Delta_R(0) : \mathsf{Im}\, z > 0\}, \quad R > 0.$$



Let γ be the oriented boundary of D consisting of the line segment γ_1 and the half-circle γ_2. We choose $R > 0$ sufficiently large such that

$$D \ni z_1, \ldots, z_m$$

which implies $W_\gamma(z_k) = 1$ for all k. The by the Cauchy's Residue Theorem,

$$\int_{\gamma_1} F(z)dz + \int_{\gamma_2} F(z)dz = 2\pi i \sum_k \mathsf{Res}_{z_k} F. \tag{14.5}$$

The half-circle integral can be estimated as

$$\left| \int_{\gamma_2} F(z)dz \right| \leq \sup_{|z|=R, \mathsf{Im}\, z \geq 0} |F(z)| \cdot L(\gamma_2)$$

$$\leq 2\pi R \sup_{|z|=R} |F(z)| = 2\pi \sup_{|z|=R} |zF(z)| \to 0, \quad R \to +\infty,$$

by the assumption (14.4). Since the right-hand side in (14.5) is independent of R, the first integral must converge to the right-hand side as $R \to +\infty$ and we obtain

$$\lim_{R\to+\infty} \int_{\gamma_1} F(z)dz = p.v. \int_{-\infty}^{\infty} F(x)dx = 2\pi i \sum_k \mathsf{Res}_{z_k} F$$

as desired. □

Example 14.4. Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx.$$

We extend the function under the integral to

$$F(z) := \frac{z^2}{1+z^4},$$

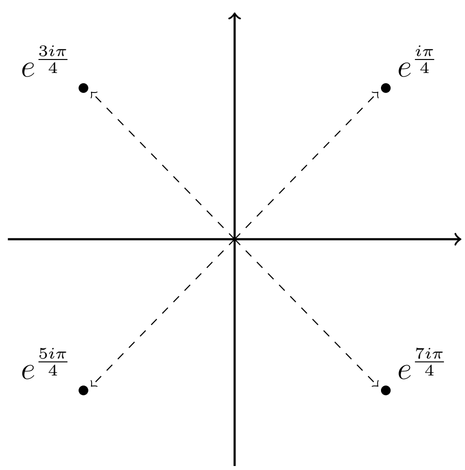
which is holomorphic everywhere except zeroes of $1+z^4$. Solving

$$1 + z^4 = 0 \iff z^4 = -1 \iff z = e^{\frac{i\pi+2i\pi k}{4}}, \quad k = 1, 2, 3, 4,$$

we find precisely two zeros

$$z_1 = e^{\frac{i\pi}{4}}, \quad z_2 = e^{\frac{3i\pi}{4}}$$

in the upper-half plane:



Then F is holomorphic in

$$U \setminus \{z_1, z_2\}, \quad z_1 = e^{\frac{i\pi}{4}},\ z_2 = e^{\frac{3i\pi}{4}}, \quad U := \mathbb{C} \setminus \{e^{\frac{5i\pi}{4}}, e^{\frac{7i\pi}{4}}\}.$$

Furthermore,

$$zF(z) = \frac{z^3}{1+z^4} \to 0, \quad z \to \infty,$$

proving the assumption (14.4) in Lemma 14.3. By the lemma,

$$p.v. \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = 2\pi i(\mathsf{Res}_{z_1} F + \mathsf{Res}_{z_2} F).$$

Finally, use Lemma 13.5 to compute the residues:

$$\mathsf{Res}_{z_1} F = \mathsf{Res}_{e^{\frac{i\pi}{4}}} \frac{z^2}{1+z^4} = \frac{z^2}{4z^3}\Big|_{z=e^{\frac{i\pi}{4}}} = \frac{1}{4z}\Big|_{z=e^{\frac{i\pi}{4}}} = \frac{e^{\frac{-i\pi}{4}}}{4} = \frac{1-i}{4\sqrt{2}},$$

$$\mathsf{Res}_{z_2} F = \mathsf{Res}_{e^{\frac{3i\pi}{4}}} \frac{z^2}{1+z^4} = \frac{z^2}{4z^3}\Big|_{z=e^{\frac{3i\pi}{4}}} = \frac{1}{4z}\Big|_{z=e^{\frac{3i\pi}{4}}} = \frac{e^{\frac{-3i\pi}{4}}}{4} = \frac{-1-i}{4\sqrt{2}},$$

and

$$I = 2\pi i \left(\frac{e^{\frac{-i\pi}{4}}}{4} + \frac{e^{\frac{-3i\pi}{4}}}{4} \right) = 2\pi i \frac{2(-i)}{4\sqrt{2}} = \frac{\pi}{\sqrt{2}}.$$

14.3. **Fourier transform.** Recall that the Fourier transform of a function $f(x)$ is given by

$$\widehat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx.$$

The following general lemma provides conditions when the Fourier transform can be calculated by means of the Residue Theorem:

Lemma 14.5. *Let U be an open set in $\mathbb{C}$,*

$$z_1, \ldots, z_m \in \{z : \mathsf{Im}\, z > 0\} \subset \{z : \mathsf{Im}\, z \geq 0\} \subset U,$$

and F a holomorphic function in $U \setminus \{z_1, \ldots, z_m\}$. Assume

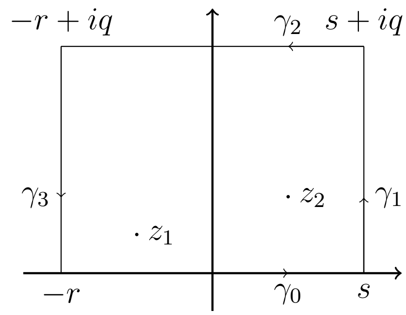
$$\lim_{z\to\infty} F(z) = 0. \tag{14.6}$$

Then for $\lambda > 0$, we have

$$\int_{-\infty}^{\infty} F(x) e^{i\lambda x} dx = 2\pi i \sum_k \mathsf{Res}_{z_k}(F(z) e^{i\lambda z}).$$

The case $\lambda < 0$ can be reduced to $\lambda > 0$ by the substitution $x \mapsto -x$.

Proof. We shall apply the Residue Theorem to the oriented boundary of a rectangle, where we denote the edges by $\gamma_0, \gamma_1, \gamma_2, \gamma_3$:



Choose r, s, q sufficiently large to ensure the rectangle contains all given points $z_1, \dots, z_m$. Then the Residue Theorem yields

$$\sum_{j=0}^{3} \int_{\gamma_j} F(z)e^{i\lambda z}dz = 2\pi i \sum_k \mathsf{Res}_{z_k}(F(z)e^{i\lambda z}). \tag{14.7}$$

Consider the integrals

$$I_j := \int_{\gamma_j} F(z)e^{i\lambda z}dz,$$

where

$$I_0 = \int_{-r}^{s} F(x)e^{i\lambda x}dx.$$

We next estimate I_1, I_2, I_3. Parametrizing $\gamma_1(t) = s + it$, $t \in [0, q]$, we calculate

$$|I_1| = \left| \int_0^q F(s+it)e^{i\lambda(s+it)}dt \right| \le \sup_{\text{image}(\gamma_1)} |F| \int_0^q e^{-\lambda t}dt$$

$$= \sup_{\text{image}(\gamma_1)} |F| \frac{1-e^{-\lambda q}}{\lambda} \le \sup_{\text{image}(\gamma_1)} |F| \frac{1}{\lambda} \to 0, \quad s \to \infty,$$

in view of the assumption (14.6). Similarly, we have

$$|I_3| \le \sup_{\text{image}(\gamma_3)} |F| \frac{1}{\lambda} \to 0, \quad r \to \infty.$$

Finally, for the top edge, we have by the basic estimate,

$$|I_2| \le \sup_{[-r+iq, s+iq]} |F(z)e^{i\lambda z}|(r+s) = \sup_{[-r+iq, s+iq]} |F| e^{-\lambda q}(r+s). \tag{14.8}$$

Now fixing $\varepsilon > 0$, for r, s sufficiently large, we have

$$|I_1| < \varepsilon/3, \quad |I_3| < \varepsilon/3.$$

Next, fixing such r, s and choosing q sufficiently large, we have by (14.8),

$$|I_2| < \varepsilon/3.$$

Substituting into (14.7), this yields

$$\left| \int_{-r}^{s} F(x)e^{i\lambda x}dx - 2\pi i \sum_{k} \mathsf{Res}_{z_k}(F(z)e^{i\lambda z}) \right| < \varepsilon$$

for s, r sufficiently large. Since $\varepsilon > 0$ is arbitrary, we obtain the desired formula for the improper integral. □

Example 14.6. Let us compute

$$I = \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2 + a^2} dx, \quad a > 0,$$

which is the Fourier transform of the function $f(x) = \frac{1}{x^2+a^2}$. The real function f is defined and continuous for all $x \in \mathbb{R}$, and extends to the complex function

$$F(z) = \frac{1}{z^2 + a^2} = \frac{1}{(z - ia)(z + ia)},$$

which is holomorphic in $\mathbb{C} \setminus \{ia, -ia\}$. Since $a > 0$, only ia is contained in the upper-half plane $\mathsf{Im}\, z > 0$.

Then we can apply Lemma 14.5 with

$$U = \mathbb{C} \setminus \{-ia\}, \quad F \in \mathcal{O}(U \setminus \{ia\}),$$

where $\mathcal{O}$ stands for holomorphic. Indeed, the assumption

$$\lim_{z \to \infty} \frac{1}{z^2 + a^2} = 0$$

is satisfied and the lemma yields

$$I = \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2 + a^2} dx = 2\pi i \mathsf{Res}_{ia} \frac{e^{i\lambda z}}{z^2 + a^2} = 2\pi i \frac{e^{i\lambda z}}{2z}\Big|_{z=ia} = \frac{\pi e^{-\lambda a}}{a}.$$

14.4. **Mellin transform.** The *Mellin transform* of a function $f(x)$ is given by

$$\{\mathcal{M}f\}(s) = \int_0^{\infty} f(x)x^{s-1}dx.$$

The following lemma demonstrates how Residue Theorem can be used to calculate the Mellin transform. A difficulty encountered here is due to the multi-valued nature of the power function $z^{s-1} = e^{(s-1)\log z}$ when extending the real power x^{s-1}.

Convention: We shall use the branch $l(z) = \ln |z| + i\varphi(z)$ of $\log z$ in $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ with $0 < \varphi(z) < 2\pi$, so that

$$\lim_{z \to x,\, \mathsf{Im}\, z > 0} l(z) = \ln x, \quad \lim_{z \to x,\, \mathsf{Im}\, z < 0} l(z) = \ln x + 2\pi i, \quad x \in \mathbb{R}_{>0}.$$

Then by a slight abuse of notation, we shall write $z^s = e^{sl(z)}$ for the branch of the power function satisfying

$$\lim_{z\to x, \operatorname{Im} z>0} z^s = x^s, \quad \lim_{z\to x, \operatorname{Im} z<0} z^s = e^{2\pi i s} x^s, \quad x \in \mathbb{R}_{>0}.$$

Lemma 14.7. *Let*

$$z_1, \ldots, z_m \notin \mathbb{R}_{\geq 0}$$

and F a holomorphic function in $\mathbb{C} \setminus \{0, z_1, \ldots, z_m\}$.

Assume $s \in \mathbb{R} \setminus \mathbb{Z}$ is such that

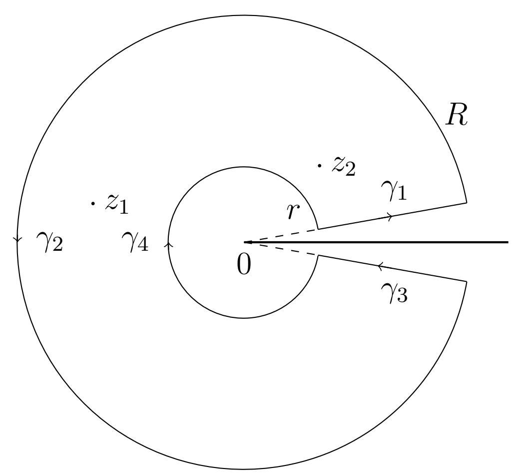
$$\lim_{z\to 0} |F(z)||z|^s = \lim_{z\to\infty} |F(z)||z|^s = 0. \tag{14.9}$$

Then we have

$$\int_0^\infty F(x)x^{s-1}dx = \frac{2\pi i}{1 - e^{2\pi i(s-1)}} \sum_k \mathsf{Res}_{z_k}(F(z)z^{s-1}).$$

Note that the choice $s \notin \mathbb{Z}$ guarantees that the denominator does not vanish.

Proof. We apply the Residue Theorem to the oriented boundary of the annulus with a sector removed as shown below:



We obtain

$$\sum_{j=1}^{4} \int_{\gamma_j} F(z)z^{s-1}dz = 2\pi i \sum_k \mathsf{Res}_{z_k}(F(z)z^{s-1}). \tag{14.10}$$

Write

$$I_j := \int_{\gamma_j} F(z)z^{s-1}dz$$

and estimate by (14.9)

$$|I_2| \leq \sup_{\text{image}(\gamma_2)} |F(z)z^{s-1}| \cdot 2\pi R = 2\pi \sup_{\text{image}(\gamma_2)} |F(z)||z|^s \to 0, \quad R \to \infty,$$

and similarly,

$$|I_4| \leq \sup_{\text{image}(\gamma_4)} |F(z)z^{s-1}| \cdot 2\pi r = 2\pi \sup_{\text{image}(\gamma_4)} |F(z)||z|^s \to 0, \quad r \to 0.$$

Fixing r and R and shrinking the sector to 0, so that both γ_2 and γ_4 approach the full circles, the path γ_1 approaches the interval $[r, R]$ and hence

$$I_1 \to \int_r^R F(x)x^{s-1}dx, \quad I_3 \to -e^{2\pi i(s-1)} \int_r^R F(x)x^{s-1}dx,$$

as the angle θ between γ_1 and γ_3 tends to 0.

Fixing $\varepsilon > 0$, we have $|I_2| < \varepsilon/2$ for R sufficiently large, and $|I_4| < \varepsilon/2$ for r sufficiently small. Then for $\theta \to 0$, (14.10) yields

$$\left| (1 - e^{2\pi i(s-1)}) \int_r^R F(x)x^{s-1}dx - 2\pi \sum_k \mathsf{Res}_{z_k}(F(z)z^{s-1}) \right| < \varepsilon,$$

and taking the limit as $r \to 0$, $R \to +\infty$,

$$(1 - e^{2\pi i(s-1)}) \int_0^\infty F(x)x^{s-1}dx = 2\pi \sum_k \mathsf{Res}_{z_k}(F(z)z^{s-1}),$$

from which the desired formula follows. □

Example 14.8. Consider the Mellin transform

$$I = \int_0^\infty \frac{x^{s-1}}{1+x}dx, \quad 0 < s < 1.$$

Then $F(z) = \frac{1}{1+z}$ is holomorphic in $\mathbb{C} \setminus \{-1\}$ and

$$\lim_{z \to 0} \frac{|z|^s}{|1+z|} = \lim_{z \to \infty} \frac{|z|^s}{|1+z|} = 0,$$

hence we can apply Lemma 14.7 to obtain

$$\begin{aligned} I &= \frac{2\pi i}{1 - e^{2\pi i(s-1)}} \mathsf{Res}_{-1} \frac{z^{s-1}}{1+z} = \frac{2\pi i}{1 - e^{2\pi i(s-1)}} (-1)^{s-1} \\ &= \frac{2\pi i}{1 - e^{2\pi i(s-1)}} e^{(s-1)l(-1)} = \frac{2\pi i}{1 - e^{2\pi i(s-1)}} e^{\pi i(s-1)} \\ &= \frac{2\pi i}{e^{-\pi i(s-1)} - e^{\pi i(s-1)}} = -\frac{\pi}{\sin \pi(s-1)}. \end{aligned}$$

15. Cauchy's Integral formula and applications

15.1. **Cauchy's Integral formula: general case.** We have previously established the Cauchy's Integral formula for polygonal sets. Now we use Cauchy's Residue Theorem to obtain the following more general version.

Theorem 15.1 (Cauchy's Integral formula). *Let $U \subset \mathbb{C}$ be open, $z_0 \in U$,*

$$f \colon U \to \mathbb{C}$$

holomorphic function, and $\gamma_1, \ldots, \gamma_n$ paths in $U \setminus \{z_0\}$ such that

$$\mathsf{Int}(\gamma_1, \ldots, \gamma_n) \subset U. \tag{15.1}$$

Then

$$\sum_j W_{\gamma_j}(z_0) f(z_0) = \frac{1}{2\pi i} \sum_j \int_{\gamma_j} \frac{f(z)dz}{z - z_0}.$$

Proof. We apply Cauchy's Residue Theorem in $U \setminus \{z_0\}$ to the function

$$g(z) = \frac{f(z)}{z - z_0},$$

which yields

$$\sum_j \int_{\gamma_j} \frac{f(z)dz}{z - z_0} = 2\pi i \sum_j W_{\gamma_j}(z_0) \mathsf{Res}_{z_0} \frac{f(z)}{z - z_0},$$

and the conclusion follows from Lemma 13.5. □

The following special case arises in many application:

Corollary 15.2 (Cauchy's Integral formula for disks). *Let $D_0, D_1, \ldots, D_n$ be open disks with*

$$\overline{D}_j \subset D_0, \quad \overline{D}_j \cap \overline{D}_k = \emptyset, \quad j \neq k, \ j, k \in \{1, \ldots, n\},$$

$U \subset \mathbb{C}$ an open set with

$$U \supset \overline{D}_0 \setminus (D_1 \cup \ldots \cup D_n)$$

and

$$f \colon U \to \mathbb{C}$$

a holomorphic function. Then

$$f(z_0) = \frac{1}{2\pi i} \left(\int_{\partial D_0} - \sum_{j=1}^{n} \int_{\partial D_j} \right) \frac{f(z)dz}{z - z_0}, \quad z_0 \in D_0 \setminus (\overline{D}_1 \cup \ldots \cup \overline{D}_n).$$

Proof. Consider the special case of Theorem 15.1, where γ_0 is the oriented boundary of D_0 and γ_j is the reverse oriented boundary of D_j for $j = 1, \ldots, n$. Since

$$\mathsf{Int}(\gamma_0, \gamma_1, \ldots, \gamma_n) = D_0 \setminus (\overline{D}_1 \cup \ldots \cup \overline{D}_n) \subset U,$$

we can apply Theorem 15.1 to obtain

$$\sum_{j=0}^{n} W_{\gamma_j}(z_0) f(z_0) = \frac{1}{2\pi i} \left(\int_{\partial D_0} - \sum_{j=1}^{n} \int_{\partial D_j} \right) \frac{f(z)dz}{z - z_0},$$

which yields the desired conclusion for

$$z_0 \in D_0 \setminus (\overline{D}_1 \cup \ldots \cup \overline{D}_n)$$

since

$$W_{\gamma_0}(z_0) = 1, \quad W_{\gamma_j}(z_0) = 0, \quad j = 1, \ldots, n.$$

□

15.2. **Mean value property.**

Corollary 15.3 (Mean value property for holomorphic functions)**.** *For every function f holomorphic in an open set containing the closed disk $\overline{\Delta_r(z_0)}$, the value at the center z_0 of the disk equals the mean value over the circle $\partial\Delta_r(z_0)$, i.e.*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt.$$

Proof. Applying Corollary 15.2 to the disk $\Delta_r(z_0)$, we obtain

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Delta_r(z_0)} \frac{f(z)dz}{z - z_0}.$$

Parametrizing $\partial\Delta_r(z_0)$ as

$$\gamma(t) = z_0 + re^{it}, \quad t \in [0, 2\pi],$$

we calculate

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} rie^{it} dt,$$

which yields the desired formula. □

15.3. **Maximum modulus principle.**

Theorem 15.4 (Local maximum modulus principle)**.** *Let f be a holomorphic function in an open set U such that the modulus function*

$$z \mapsto |f(z)|$$

achieves a local maximum at z_0. Then f is constant in some disk $\Delta_\varepsilon(z_0)$, $\varepsilon > 0$.

Proof. Choose any $\varepsilon > 0$ such that $\overline{\Delta_\varepsilon(z_0)} \subset U$. Then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \varepsilon e^{it})dt$$

by the mean value property, which implies

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \varepsilon e^{it})|dt$$

$$\Longrightarrow \frac{1}{2\pi}\int_0^{2\pi} (|f(z_0)| - |f(z_0 + \varepsilon e^{it})|)dt \le 0. \tag{15.2}$$

On the other hand, if z_0 is a local maximum for $|f(z)|$, then

$$|f(z_0)| - |f(z_0 + \varepsilon e^{it})| \ge 0, \quad t \in [0, 2\pi], \tag{15.3}$$

for any sufficiently small $\varepsilon > 0$. Since f is continuous, both (15.2) and (15.3) can only hold if

$$|f(z_0)| = |f(z_0 + \varepsilon e^{it})|, \quad t \in [0, 2\pi],$$

i.e. $|f(z)| \equiv \text{const}$ in $\Delta_\varepsilon(z_0)$.

If $|f(z)| \equiv c = 0$, then $f \equiv 0$ as desired. Otherwise, $c := f(z_0) \neq 0$, and

$$g(z) := \mathsf{Log}(\frac{1}{c}f(z)) = \ln\frac{|f(z)|}{c} + i\mathsf{Arg}\frac{1}{c}f(z) = i\mathsf{Arg}\frac{1}{c}f(z)$$

is a well-defined local branch of $\log(\frac{1}{c}f(z))$ in $\Delta_\varepsilon(z_0)$ for $\varepsilon > 0$ sufficiently small. Then $g(z)$ is a holomorphic function with $\mathsf{Re}\, g \equiv 0$, hence also $\mathsf{Im}\, g \equiv \text{const}$ by the Cauchy-Riemann equations. Therefore $g \equiv \text{const}$ implying

$$f(z) = ce^{g(z)} \equiv \text{const}$$

as desired. □

15.4. **Power series expansion.**

Definition 15.5. A *power series* centered at z_0 is any series of the form

$$\sum_{n\geq 0} c_n(z-z_0)^n, \quad c_n \in \mathbb{C}.$$

Theorem 15.6. *Let f be a holomorphic function in a disk*

$$\Delta_R(z_0), \quad 0 < R \leq +\infty.$$

Then f has the power series expansion

$$f(z) = \sum_{n\geq 0} c_n(z-z_0)^n, \quad z \in \Delta_R(z_0),$$

where

$$c_n = \frac{1}{2\pi i}\int_{\partial\Delta_r(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}dz, \quad 0 < r < R, \quad n = 0, 1, \ldots.$$

Proof. By Cauchy's integral formula for the disk $\partial\Delta_r(z_0)$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta_r(z_0)} \frac{f(w)dw}{w-z}, \quad z \in \Delta_r(z_0). \tag{15.4}$$

The idea is to expand in a power series centered at z_0 the function under the integral for each fixed w. For this, write

$$\frac{1}{w-z} = \frac{1}{w-z_0} \cdot \frac{1}{1-\frac{z-z_0}{w-z_0}} = \sum_{n\geq 0} \frac{(z-z_0)^n}{(w-z_0)^{n+1}}, \tag{15.5}$$

for which we have the majorant series

$$\sum_{n\geq 0} \left|\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\right|$$

that converges since

$$z \in \Delta_r(z_0), w \in \partial\Delta_r(z_0) \implies \left|\frac{z-z_0}{w-z_0}\right| = \frac{|z-z_0|}{r} < 1.$$

Since also f is bounded on the compact set $\partial D_r(z_0)$,

$$\left|\frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}}\right| \leq \frac{\sup_{\partial\Delta_r(z_0)}|f|}{r} \cdot \left|\frac{z-z_0}{r}\right|^n,$$

the Weierstrass M-test implies that

$$\sum_{n\geq 0} \frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}}$$

converges uniformly for $w \in \partial\Delta_r(z_0)$ and $z \in \Delta_r(z_0)$ fixed. Then the basic estimate for the integral implies that the sum of the integrals of summands converges to the integral of the sum:

$$\int_{\partial\Delta_r(0)} \sum_{n\geq 0} \frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}} dw = \sum_{n\geq 0} \int_{\partial\Delta_r(0)} \frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}} dw,$$

and (15.4) becomes

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta_r(0)} \sum_{n\geq 0} \frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}} dw$$

$$= \sum_{n\geq 0} \left(\frac{1}{2\pi i} \int_{\partial\Delta_r(0)} \frac{f(w)}{(w-z_0)^{n+1}} dw \right) (z-z_0)^n = \sum_{n\geq 0} c_n (z-z_0)^n,$$

where the coefficients

$$c_n = \frac{1}{2\pi i} \int_{\partial\Delta_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} dw, \quad n = 0, 1, \ldots, \tag{15.6}$$

are as in the theorem. This shows the power series expansion for $z \in \Delta_r(z_0)$. However, for every $z \in \Delta_R(z_0)$, there exists $0 < r < R$ with $z \in \Delta_r(z_0)$. Since the right-hand side in (15.6) is independent of r by Cauchy's theorem, the power series expansion

$$f(z) = \sum_{n\geq 0} c_n (z-z_0)^n$$

holds for all $z \in \Delta_R(z_0)$ as stated. □

15.5. **Laurent series expansion.** A Laurent series is a generalization of a power series with both negative and positive powers:

Definition 15.7 (Laurent series)**.** A *Laurent series* centered at z_0 is any series of the form

$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n, \quad c_n \in \mathbb{C}. \tag{15.7}$$

Convergence for Laurent series is defined by taking separately parts with positive and negative powers:

Definition 15.8. A Laurent series (15.7) is said to be convergent if both series

$$\sum_{n\geq 0} c_n (z-z_0)^n, \quad \sum_{n<0} c_n (z-z_0)^n \tag{15.8}$$

are convergent, in which case the sum of both limits is by definition the sum of the Laurent series (15.7).

Similarly to power series expansions for holomorphic functions in disks, we have Laurent series expansions for holomorphic functions in rings:

Theorem 15.9. *Let f be a holomorphic function in a ring*

$$\Delta_R(z_0) \setminus \overline{\Delta_r(z_0)} = \{z \in \mathbb{C} : r < |z - z_0| < R\}, \quad 0 \le r < R \le +\infty.$$

Then f has the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad z \in \Delta_R(z_0) \setminus \overline{\Delta_r(z_0)},$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial \Delta_t(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad r < t < R, \quad n \in \mathbb{Z}. \tag{15.9}$$

Proof. Considers disks

$$\Delta_{r'}(z_0), \quad \Delta_{R'}(z_0), \quad r < r' < R' < R,$$

so that

$$\overline{\Delta_{R'}(z_0)} \setminus \Delta_{r'}(z_0) \subset U := \Delta_R(z_0) \setminus \overline{\Delta_r(z_0)}$$

Then by Cauchy's integral formula in the form of Corollary 15.2,

$$f(z) = \frac{1}{2\pi i} \left(\int_{\partial \Delta_{R'}(z_0)} - \int_{\partial \Delta_{r'}(z_0)} \right) \frac{f(w) dw}{w - z}, \quad z \in \Delta_{R'}(z_0) \setminus \Delta_{r'}(z_0). \tag{15.10}$$

We next follows the argument of the of Theorem 15.6 for each integral. For the first integral, we consider, as before, the expansion

$$\frac{f(w)}{w - z} = \frac{f(w)}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}} = f(w) \sum_{n \ge 0} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}, \tag{15.11}$$

which converges for

$$|z - z_0| < |w - z_0| = R'$$

since

$$\left| \frac{z - z_0}{w - z_0} \right| = \frac{|z - z_0|}{R'} < 1.$$

Then, as before, the Weierstrass M-test implies that

$$\sum_{n \ge 0} \frac{f(w)(z - z_0)^n}{(w - z_0)^{n+1}}$$

converges uniformly for $w \in \partial\Delta_{R'}(z_0)$ and $z \in \Delta_{R'}(z_0)$ fixed, and by the basic estimate for the integral, the sum of the integrals of summands converges to the integral of the sum:

$$\int_{\partial\Delta_{R'}(0)} \sum_{n\geq 0} \frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}} dw = \sum_{n\geq 0} \int_{\partial\Delta_{R'}(0)} \frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}} dw,$$

and we obtain the expansion of the first integral in (15.10):

$$\frac{1}{2\pi i} \int_{\partial\Delta_{R'}(0)} \frac{f(w)}{w-z} dw = \sum_{n\geq 0} c_n (z-z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial\Delta_{R'}(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} dw. \tag{15.12}$$

On the other hand, for the second integral over $\partial\Delta_{r'}(z_0)$, we have the inequality

$$|z-z_0| > |w-z_0| = r', \tag{15.13}$$

hence consider the expansion

$$\frac{f(w)}{w-z} = -\frac{f(w)}{z-z_0} \cdot \frac{1}{1-\frac{w-z_0}{z-z_0}} = -f(w) \sum_{n\geq 0} \frac{(w-z_0)^n}{(z-z_0)^{n+1}}, \tag{15.14}$$

which converges since (15.13) implies

$$\left| \frac{w-z_0}{z-z_0} \right| = \frac{r'}{|z-z_0|} < 1.$$

Again, using the boundedness of f on $\partial\Delta_{r'}(z_0)$ and the Weierstrass M-test,

$$\int_{\partial\Delta_{r'}(0)} \sum_{n\geq 0} \frac{f(w)(w-z_0)^n}{(z-z_0)^{n+1}} dw = \sum_{n\geq 0} \int_{\partial\Delta_{r'}(0)} \frac{f(w)(w-z_0)^n}{(z-z_0)^{n+1}} dw$$

and we obtain the expansion of the second integral in (15.10):

$$\frac{1}{2\pi i} \int_{\partial\Delta_{r'}(0)} \frac{f(w)}{w-z} dw = -\sum_{n\geq 0} c_{-(n+1)} (z-z_0)^{-(n+1)},$$

where

$$c_{-(n+1)} = \frac{1}{2\pi i} \int_{\partial\Delta_{r'}(z_0)} \frac{f(w)}{(w-z_0)^{-n}} dw, \quad n = 0, 1, \ldots. \tag{15.15}$$

Finally, substituting both integrals into (15.10), we obtain

$$f(z) = \sum_{m=-\infty}^{\infty} c_m (z - z_0)^m, \quad r' < |z - z_0| < R', \tag{15.16}$$

where all coefficients satisfy

$$c_m = \frac{1}{2\pi i} \int_{\partial \Delta_t(z_0)} \frac{f(z)}{(z - z_0)^{m+1}} dz, \quad m \in \mathbb{Z},$$

for all $r < t < R$, since these integrals are independent of t by Cauchy's theorem.

We conclude the proof by observing that for every

$$z \in \Delta_R(z_0) \setminus \overline{\Delta_r(z_0)},$$

we can choose $r < r' < R' < R$ such that $r' < |z - z_0| < R'$ implying the expansion (15.16), which therefore holds for all z in the ring. □

16. Properties of power and Laurent series

For simplicity we shall consider power series centered at 0. The general case of arbitrary center can be reduced to the center 0 by translation.

16.1. **Abel's lemma and applications.**

Lemma 16.1 (Abel's lemma). *Assume that a power series*

$$\sum_{n=0}^{\infty} c_n z^n \tag{16.1}$$

converges for $z = z_0$*. Then*

1. *the series* (16.1) *converges absolutely for every* z *with* $|z| < |z_0|$*;*
2. *the series* (16.1) *converges uniformly on every disk* $\Delta_r(0)$ *with* $r < |z_0|$.

Proof. Since (16.1) converges for $z = z_0$, by the nth term test,

$$c_n z_0^n \to 0, \quad n \to \infty.$$

In particular, the sequence $(c_n z_0^n)$ is bounded, i.e. there exists $M > 0$ with

$$|c_n z_0^n| \le M, \quad n = 0, 1, \dots.$$

Then for $|z| \le r < |z_0|$,

$$|c_n z^n| = |c_n z_0^n| \cdot \left| \frac{z}{z_0} \right|^n \le M \left(\frac{r}{|z_0|} \right)^n.$$

Since $r < |z_0|$, the series

$$\sum_{n\ge 0} M\left(\frac{r}{|z_0|}\right)^n$$

converges and (1) and (2) follow respectively from the comparison and Weierstrass M-test. ☐

Corollary 16.2 (Uniform convergence of Laurent series)**.** *Let*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \quad 0 \le r < |z| < R \le \infty,$$

be a Laurent series expansion in the ring $\Delta_R(0)\setminus\overline{\Delta_r(0)}$*. Then the Laurent series converges uniformly in every strictly smaller closed ring*

$$\overline{\Delta_{R'}(0)} \setminus \Delta_{r'}(0), \quad r < r' < R' < R.$$

Proof. Recall that the convergence of the Laurent series $\sum_{n=-\infty}^{\infty} c_n z^n$ means that both positive and negative parts

$$\sum_{n\ge 0} c_n z^n, \quad \sum_{n<0} c_n z^n = \sum_{m>0} c_{-m}\left(\frac{1}{z}\right)^m.$$

Then Abel's lemma, part (2), implies that the positive part converges uniformly for $|z| \le R'$, $R' < R$, and the negative part converges uniformly for $|1/z| < 1/r'$, i.e. for $|z| > r'$. Then both positive and negative parts converge in the smaller closed ring as desired. ☐

16.2. **Laurent series and residues.** The following result provides a general method of calculating residues.

Theorem 16.3. *Let*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n, \quad 0 < |z-z_0| < R \le +\infty,$$

be a Laurent series expansion in the ring $\Delta_R(z_0)\setminus\{z_0\}$*. Then the residue of* f *equals to the coefficient of* $(z-z_0)^{-1}$*:*

$$\mathsf{Res}_{z_0} f = c_{-1}.$$

Proof. In view of the uniform convergence in every smaller ring by Corollary 16.2, we can calculate the residue by integrating the Laurent series expansion term-wise:

$$\mathsf{Res}_{z_0} f = \frac{1}{2\pi i}\int_{\partial\Delta_t(z_0)} f(z)dz = \frac{1}{2\pi i}\sum_{n\in\mathbb{Z}}\int_{\partial\Delta_t(z_0)} c_n(z-z_0)^n dz, \quad 0 < t < R.$$

Each power function $p_n(z) = (z - z_0)^n$ for $n \neq -1$ has the antiderivative

$$P_n(z) = \frac{(z - z_0)^{n+1}}{n + 1},$$

hence each respective integral vanishes. The only remaining integral is

$$\frac{1}{2\pi i} \int_{\partial \Delta_t(z_0)} \frac{c_{-1}}{z - z_0} dz = c_{-1},$$

proving the desired formula. □

16.3. **Radius of convergence.**

Definition 16.4. The *radius of convergence* of the power series

$$\sum_{n \geq 0} c_n z^n$$

is given by

$$R := \sup\{|z| : \sum c_n z^n \text{ converges}\}, \quad 0 \leq R \leq +\infty.$$

Corollary 16.5. *Let R be the radius of convergence of the power series*

$$\sum_{n \geq 0} c_n z^n.$$

Then:

(1) *the series converges absolutely for $|z| < R$;*
(2) *the series diverges for $|z| > R$.*

Proof. Let z satisfy $|z| < R$. By Definition 16.4, there exists z_0 such that $|z| < |z_0| < R$ and $\sum c_n z_0^n$ converges. Then (1) follows from Abel's lemma.

Now let z satisfy $|z| > R$. Suppose by contradiction that the series converges at z. Then by Abel's lemma, $\sum_n c_n w^n$ converges for $|w| < |z|$. In particular, we can find w with $R < |w| < |z|$ where the series converges, which contradicts Definition 16.4. □

Example 16.6. The geometric series

$$\sum_n z^n$$

converges for $|z| < 1$ and diverges for $|z| \geq 1$. Hence its radius of convergence is 1.

Example 16.7. The series

$$\sum_n \frac{z^n}{n!}$$

is convergent for all $z \in \mathbb{C}$ by the ratio test, since

$$\left| \frac{z^{n+1}}{(n+1)!} / \frac{z^n}{n!} \right| = \frac{|z|}{n+1} \to 0, \quad n \to \infty.$$

Hence its radius of convergence is $+\infty$.

Example 16.8. The series

$$\sum_n n! z^n$$

is divergent for all $z \neq 0$ by the nth term test, since

$$|n! z^n| \to \infty, \quad n \to \infty.$$

Hence its radius of convergence is 0.

16.4. **Differentiation of power series.**

Theorem 16.9. *Let*

$$\sum_n c_n z^n$$

be a power series with radius of convergence $R > 0$. Then for $|z| < R$, the sum of the series is $\mathbb{C}$-differentiable with the derivative obtain by term-wise differentiation:

$$\left(\sum_{n \geq 0} c_n z^n \right)' = \sum_{n \geq 1} n c_n z^{n-1}. \tag{16.2}$$

Proof. We first prove that the power series of the term-wise derivatives

$$\sum_{n \geq 1} n c_n z^{n-1}$$

converges for $|z| < R$. For this, choose any z_0 with

$$|z| < |z_0| < R,$$

so that

$$\sum_n c_n z_0^n$$

converges absolutely. Then

$$|n c_n z^{n-1}| \leq \frac{n}{|z_0|} \left| \frac{z}{z_0} \right|^{n-1} |c_n z_0^n|.$$

Since

$$\frac{n}{|z_0|}\left|\frac{z}{z_0}\right|^{n-1} \to 0, \quad n \to \infty,$$

the series of the term-wise derivatives converges absolutely by the comparison test.

Next for any $|z|, |z_0| < r < R$, we estimate the increments of powers as

$$(16.3)\quad |z^n - z_0^n| = |z - z_0||z^{n-1} + z^{n-2}z_0 + \ldots + z_0^{n-1}| \le |z - z_0| \cdot nr^{n-1}.$$

Then comparing their ratios with the nth term derivatives and using (16.3), we obtain

$$\left|\frac{z^n - z_0^n}{z - z_0} - nz_0^{n-1}\right| = \left|z^{n-1} + z^{n-2}z_0 + \ldots + zz_0^{n-2} + z_0^{n-1} - nz_0^{n-1}\right|$$
$$\le |z^{n-1} - z_0^{n-1}| + |z^{n-2}z_0 - z_0^{n-1}| + \ldots + |zz_0^{n-2} - z_0^{n-1}| + |z_0^{n-1} - z_0^{n-1}|$$
$$\le |z^{n-1} - z_0^{n-1}| + |z_0||z^{n-2} - z_0^{n-2}| + \ldots + |z_0^{n-2}||z - z_0| + |z_0^{n-1}||1 - 1|$$
$$\le |z - z_0| \cdot n(n-1)r^{n-1},$$

and therefore

$$(16.4)\quad \left|\sum_n c_n \frac{z^n - z_0^n}{z - z_0} - \sum_n c_n n z_0^{n-1}\right| \le |z - z_0| \sum_n n(n-1)|c_n| r^{n-1}.$$

We have proved that the series with term-wise derivatives

$$\sum nc_n z^{n-1}$$

converges absolutely in the same disk $|z| < R$, hence also the series with twice term-wise derivatives

$$\sum_n n(n-1)c_n z^{n-2} = \frac{1}{z}\sum_n n(n-1)c_n z^{n-1}.$$

Since $r < R$, this shows that the series

$$\sum_n n(n-1)c_n r^{n-1}$$

converges. Then the rigth-hand side of (16.4) converges to 0 as $z \to z_0$, hence the left-hand side does, proving the desired conclusion. □

Combining with the power series expansion for holomorphic functions, we obtain the following important result:

Corollary 16.10. *Let f be holomorphic in an open set $U \subset \mathbb{C}$. Then the derivative f' is again holomorphic in U. In particular, f is infinitely $\mathbb{C}$-differentiable.*

Proof. Since U is open, for every $z_0 \in U$, there exists a disk $\Delta_r(z_0) \subset U$. By Theorem 15.6, f has a power series expansion

$$f(z) = \sum_{n\geq 0} c_n (z-z_0)^n, \quad z \in \Delta_r(z_0).$$

Then by Theorem 16.9,

$$f'(z) = \sum_{n\geq 0} n c_n (z-z_0)^{n-1},$$

which is another convergent power series. Applying Theorem 16.9 again, we conclude that f' is holomorphic in $\Delta_r(z_0)$. Since z_0 is arbitrary, this shows that f' is holomorphic in U as desired. $\square$

16.5. **Morera's theorem.**

Theorem 16.11 (Morera)**.** *Let $U \subset \mathbb{C}$ be open and $f\colon U \to \mathbb{C}$ a continuous function satisfying*

$$\int_{\partial T} f(z)dz = 0$$

for every closed triangle $T \subset U$. Then f is holomorphic in U.

Proof. Let $z_0 \in U$ with a disk $D := \Delta_r(z_0) \subset U$. Then D is star-shaped and by Lemma 12.5, the restriction $f|_D$ has an antiderivative $F\colon D \to \mathbb{C}$, i.e. $F' = f|_D$. Since F is holomorphic, it is infinitely $\mathbb{C}$-differentiable by Corollary 16.10. In particular, f_D is also $\mathbb{C}$-differentiable. Since the disk is arbitrary, f is holomorphic as desired. $\square$

Morera's theorem can be seen as a converse of the Cauchy-Goursat theorem. Corollary 16.10 is the last missing ingredient we need to prove it.

16.6. **Taylor's formula.** Another important consequence of Theorem 16.9 on differentiation of power series is the Taylor's formula for the coefficients:

Corollary 16.12 (Taylor's formula)**.** *Let*

$$f(z) = \sum_{n\geq 0} c_n (z-z_0)^n, \quad z \in \Delta_r(z_0), \quad r > 0.$$

be a convergent power series. Then

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

In particular, the coefficients in any power series expansion of f are uniquely determined by f.

Proof. By Theorem 16.9, calculating the mth derivative by term-wise differentiation,

$$f^{(m)}(z_0) = \sum_{n\geq m} n(n-1)\dots(n-m+1)c_n(z-z_0)^{n-m}|_{z=z_0} = m!c_m.$$

□

16.7. **Examples of power series expansions.** We use Taylor's formula to calculate some important examples of power series expansions.

Example 16.13. The exponential function $f(z) = e^z$ is holomorphic in $\mathbb{C}$ and satisfies $f^{(n)} = e^z$. Hence by the power series expansion $e^z = \sum_n c_n z^n$ for all z with

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{e^z}{n!}\bigg|_{z=0} = \frac{1}{n!} \implies e^z = \sum_{n\geq 0} c_n z^n = \sum_{n\geq 0} \frac{z^n}{n!}.$$

Example 16.14. Similarly, calculating the nth derivatives of $\cos z$ and $\sin z$ and using Taylor's formula, we obtain their expansions

$$\cos z = \sum_{k\geq 0} \frac{(-1)^k z^{2k}}{(2k)!}, \quad \sin z = \sum_{k\geq 0} \frac{(-1)^k z^{2k+1}}{(2k+1)!}.$$

Example 16.15. Let $f(z) = \mathsf{Log}(z+1)$, where Log is the principal branch of $\log z$. Then f is holomorphic in $U = \mathbb{C} \setminus \mathbb{R}_{\leq -1}$, where $\Delta_1(0) \subset U$ is the maximal disk with center 0, where f has a power series expansion

$$f(z) = \sum c_n z^n.$$

Then $c_0 = f(0) = \mathsf{Log}1 = 0$ and differentiating we obtain

$$\frac{1}{z+1} = \sum_{n\geq 1} nc_n z^{n-1}, \quad |z| < 1.$$

On the other hand,

$$\frac{1}{z+1} = \sum_{n\geq 1} (-z)^{n-1}, \quad |z| < 1.$$

By Taylor's formula, the power series coefficients are determined by the sum of the series, hence

$$nc_n = (-1)^{n-1} \implies c_n = \frac{(-1)^{n-1}}{n}$$

and we obtain the power series expansion of the logarithm function

$$\mathsf{Log}(1+z) = \sum_{n\geq 1} \frac{(-1)^{n-1} z^n}{n}.$$

We could have used Taylor's formula directly by calculating all higher order derivative of f but the present method is somewhat shorter.

16.8. **Cauchy's estimates.** Cauchy's estimates are important estimates for the derivatives of holomorphic functions:

Theorem 16.16 (Cauchy's estimates)**.** *Let f be holomorphic in a disk $\Delta_R(z_0)$. Then*

$$|f^{(n)}(z_0)| \leq \frac{n! \sup |f|}{R^n}, \quad n = 1, 2, \ldots.$$

Proof. By Theorem 15.6 on power series expansion,

$$f(z) = \sum_{n \geq 0} c_n (z - z_0)^n, \quad z \in \Delta_R(z_0),$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial \Delta_r(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad 0 < r < R.$$

Estimating the integrals, we obtain

$$|c_n| \leq \frac{1}{2\pi} \frac{\sup |f|}{r^{n+1}} 2\pi r = \frac{\sup |f|}{r^n}.$$

Since r is arbitrary with $0 < r < R$, taking the limit as $r \to R$, we obtain

$$|c_n| \leq \frac{\sup |f|}{R^n}.$$

Now the estimate for the nth derivative follows from Taylor's formula:

$$|f^{(n)}(z_0)| = |n! c_n| \leq \frac{n! \sup |f|}{R^n}$$

as desired. □

16.9. **Liouville's theorem.**

Theorem 16.17 (Liouville's theorem)**.** *Let f be a holomorphic function in $\mathbb{C}$, which is also bounded. Then $f = \text{const}$.*

Proof. Since f is bounded in $\mathbb{C}$,

$$M := \sup_{z \in \mathbb{C}} |f(z)| < +\infty.$$

Since f is holomorphic in any disk $\Delta_R(0)$, Cauchy's estimates yield

$$|f^{(n)}(0)| \leq \frac{n! \sup |f|}{R^n} \leq \frac{n! M}{R^n}, \quad n = 1, 2, \ldots.$$

Since R is arbitrary, letting $R \to \infty$ we obtain

$$|f^{(n)}(0)| = 0 \implies f^{(n)}(0) = 0, \quad n = 1, 2, \ldots,$$

and the conclusion follows from the power series expansion and Taylor's formula:

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} z^n = f(0).$$

□

Corollary 16.18 (Fundametal theorem of algebra)**.** *Let*

$$P(z) = a_n z^n + \ldots + a_0, \quad a_n \neq 0, \quad n > 0,$$

be a polynomial of a positive degree. Then P has a root in $\mathbb{C}$, i.e. there exists $z \in \mathbb{C}$ with $P(z) = 0$.

Proof. Assume by contradiction that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then

$$f(z) := \frac{1}{P(z)}$$

is defined and holomorphic in $\mathbb{C}$. Factoring out the dominant term,

$$|f(z)| = \frac{1}{|P(z)|} = \frac{1}{|a_n||z|^n} \frac{1}{1 + \frac{a_1}{a_n z^{n-1}} + \ldots + \frac{a_0}{a_n z^n}},$$

we see that

$$|f(z)| \to 0, \quad z \to \infty.$$

In particular, f is bounded in $\mathbb{C} \setminus \Delta_R(0)$ for some $R > 0$. On the other hand, f is continuous on the compact set $\overline{\Delta_R(0)}$, hence is also bounded on $\overline{\Delta_R(0)}$. Then f is bounded on $\mathbb{C}$ and therefore, by the Liouville's theorem, is constant. But then also $P(z) = 1/f(z)$ is constant, which contradicts the assumptions, proving the claim. □

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