Exercise 1
Let \( p_1, p_2, p_3, p_4 \) be distinct points of \( \mathbb{P}^2 \) with no 3 collinear.

(i) Prove that there exists unique coordinate system in which the 4 points are \((1, 0, 0), (0, 1, 0), (0, 0, 1) \) and \((1, 1, 1)\).

(ii) Find all conics passing through \( p_1, \ldots, p_4 \) and \( p_5 = (a, b, c) \).

Exercise 2
Use the parametrization of the cuspidal cubic \( C = \{y^2 = x^3\} \) to show that any polynomial vanishing on \( C \) is divisible by \( y^2 - x^3 \).

Exercise 3
Let \( C \) be the curve given by \( f(x, y) = 0 \) and \( p = (a, b) \in C \). Assume that the gradient \( \nabla f = (f_x, f_y) \) is nonzero at \( (a, b) \).

(i) Show that the equation

\[
f_x(p)(x - a) + f_y(p)(y - b) = 0
\]

defines the unique tangent line to \( C \) at \( p \), i.e. the unique line \( L \) such that \( f|_L \) has a multiple root at \( p \).

(ii) Show that the tangent line can be obtained as the limit of the secant line passing through \( p \) and another point \( q \in C \) as \( q \to p \). Hint. Write an equation for the secant, similar to the tangent line equation, where the derivatives are replaced by the corresponding increment ratios.

Exercise 4
Let \( C \) be given by \( y^2 = x(x - 1)(x + 1) \).

(i) What is the projectivization \( \tilde{C} \) of \( C \), i.e. the cubic obtained in \( \mathbb{P}^2 \) by homogenization of the equation for \( C \)?

(ii) Find the point at infinity of \( \tilde{C} \).

(iii) Chosing the origin at infinity, find all points of order 2 with respect to the group law, i.e. all points \( A \in \tilde{C} \) with \( 2A = A + A = 0 \). Hint. Use Exercise 3.