Course 214 2009 Complex Analysis)

Sheet 3

Due: at the end of the lecture

Sheet 2, Exercise 5

Find all z, for which the following identity holds:

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

Solution
We have

$$\sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}} = \frac{1}{4^2} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \frac{1}{4^2} \frac{1}{1 - z/4}$$

for $|z/4| < 1$, i.e. $|z| < 4$. Substituting in the above identity we conclude that the latter holds for $0 < |z| < 4$. On the other hand, for $z = 0$ both sides are not defined and for $|z| \geq 4$, the series diverges. Hence the identity holds precisely for $0 < |z| < 4$.

Sheet 2, Exercise 6

Let $f(z)$ be any branch of $\mathsf{log} z$ defined on an open set. Show that f is holomorphic and $f'(z) = \frac{1}{z}$.

Solution Since any branch $w = f(z)$ of $\mathsf{log} z$ is continuous and is an inverse of $z = e^w$, it is holomorphic and

$$f'(z) = \frac{1}{(e^w)'} = \frac{1}{e^w} = \frac{1}{z}.$$

Sheet 3, Exercise 3

Evaluate the integrals:

(i) $\int_{|z|=2} \frac{1}{z(z-1)(z-3)}\, dz$;

Solution

The function $\frac{1}{z(z-1)(z-3)}$ is holomorphic away from the singularities at $0, 1, 3$, of which 0 and 1 are inside the circle $|z| = 2$. By the Residue Theorem,

$$I = \int_{|z|=2} \frac{1}{z(z-1)(z-3)}\, dz = 2\pi i \left(\mathrm{Res}_0 \frac{1}{z(z-1)(z-3)} + \mathrm{Res}_1 \frac{1}{z(z-1)(z-3)} \right).$$

We use the formula

$$(*) \qquad \mathrm{Res}_a \frac{g(z)}{z-a} = g(a).$$

Then

$$I = 2\pi i \left(\frac{1}{(0-1)(0-3)} + \frac{1}{1(1-3)} \right) = -\frac{\pi i}{3}.$$

(ii) $\int_0^{2\pi} \frac{\sin\theta}{2+\cos\theta} d\theta$;

Solution Using the substitution $z = e^{i\theta}$, $\cos\theta = \frac{z+z^{-1}}{2}$, $\sin\theta = \frac{z-z^{-1}}{2i}$, $d\theta = \frac{dz}{iz}$, we obtain

$$I = \int_0^{2\pi} \frac{\sin\theta}{2+\cos\theta} d\theta = \int_{|z|=1} \frac{\frac{z-z^{-1}}{2i}}{2+\frac{z+z^{-1}}{2}} \frac{dz}{iz} = \int_{|z|=1} \frac{1-z^2}{z(z^2+4z+1)}\, dz.$$

The function $\frac{1-z^2}{z(z^2+4z+1)}$ is holomorphic away from the singularities at 0 and $-2 \pm \sqrt{3}$, of which 0 and $-2+\sqrt{3}$ are inside the circle $|z| = 1$. By the Residue Theorem,

$$I = 2\pi i \left(\mathrm{Res}_0 \frac{1-z^2}{z(z^2+4z+1)} + \mathrm{Res}_{(-2+\sqrt{3})} \frac{1-z^2}{z(z^2+4z+1)} \right).$$

We use the formula (*) and

$$(**) \qquad \mathrm{Res}_a \frac{g(z)}{h(z)} = \frac{g(a)}{h'(a)}.$$

Then

$$I = 2\pi i \left(\frac{1-0^2}{0^2+4\cdot 0+1} + \frac{1-(-2+\sqrt{3})^2}{(-2+\sqrt{3})(2(-2+\sqrt{3})+4)} \right) = 0.$$

(iii) $\int_{-\infty}^{\infty} \frac{x^2}{x^4-2x^2+2}\, dx$;

Solution

Use the extension $F(z) = \frac{z^2}{z^4-2z^2+2}$, which satisfies

$$\lim_{z\to\infty} zF(z) = 0$$

and is holomorphic away from the singularities

$$\pm 2^{1/4} e^{\pm i\frac{\pi}{8}},$$

of which $2^{1/4}e^{i\frac{\pi}{8}}$ and $-2^{1/4}e^{-i\frac{\pi}{8}}$ are in the upper half-plane $\{\mathrm{Im} z > 0\}$. Then using the Residue Theorem, we obtain

$$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4 - 2x^2 + 2}\, dx = 2\pi i \left(\mathrm{Res}_{(2^{1/4}e^{i\frac{\pi}{8}})} F(z) + \mathrm{Res}_{(-2^{1/4}e^{-i\frac{\pi}{8}})} F(z) \right).$$

Using (**) we have

$$\mathrm{Res}_a \frac{z^2}{z^4 - 2z^2 + 2} = \frac{a^2}{4a^3 - 4a} = \frac{a}{4(a^2-1)}$$

and hence

$$I = 2\pi i \left(\frac{2^{1/4}e^{i\frac{\pi}{8}}}{4(\sqrt{2}e^{i\frac{\pi}{4}} - 1)} + \frac{-2^{1/4}e^{-i\frac{\pi}{8}}}{4(\sqrt{2}e^{-i\frac{\pi}{4}} - 1)} \right) = 2^{1/4}\pi \cos\frac{\pi}{8}.$$