MA1214

Introduction to group theory

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Solutions to Sheet 4

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1. (a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} = (2\ 4\ 5)(1\ 3)$$

The 3-cycle is composed of 2 transpositions. Thus the overall sign is -1.

(b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 2 & 6 & 4 \end{pmatrix} = (2\ 5\ 6\ 4)(1\ 3)$$

The 4-cycle is composed of 3 transpositions. Thus the overall sign is $+1$.

(c)

$$\begin{aligned}(12)(234)(3456) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 & 3 \\ 1 & 3 & 2 & 5 & 6 & 4 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix} \\ &= (4\ 5\ 6)(1\ 2\ 3)\end{aligned}$$

The sign is $+1$.

Note: In general, an n-cycle is composed of $(n-1)$ transpositions. (In order to detemine the sign of a permutation, rewrite it as a composition of disjoint cycles.)

2. When the problem asks you to *find* a solution, some explanation is required.

(a) $(S, \circ)$ with $S = \{(1), (1\ 2)\}$, and $\circ$ is the composition of permutations. Indeed, $(1, 2)$ is an element of order 2:

$$(1\ 2) \circ (1\ 2) = e$$

(b) $(S, \circ)$ with $S = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$.
$(1 2 3)$ is an element of order 3: Indeed, repeated application of $(1 2 3)$ yields

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

(to be read as a composition of permutations). It generates the cycles $(1\ 2\ 3)^2 = (1\ 3\ 2) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and the identity $e = (1) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.

(c) $S = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$.
Indeed, we already know that S must contain $(1), (1\ 2), (1\ 2\ 3), (1\ 3\ 2)$. We have

$$\begin{aligned}
&(1) = (1\ 2)^2 = (1\ 2\ 3)^3 = (1\ 2\ 3)(1\ 3\ 2) = (1\ 3\ 2)(1\ 2\ 3)\\
&(1\ 2)\\
&(1\ 2\ 3) = (1\ 3\ 2)^2 \quad (= (1\ 2\ 3)^4)\\
&(1\ 2\ 3)^2 = (1\ 3\ 2)
\end{aligned}$$

Mixed products of $(1\ 2)$ and $(1\ 2\ 3)$ (and powers thereof):

$$(1\ 2\ 3)(1\ 2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\ 3) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} = (1\ 2)(1\ 3\ 2)$$

$$(1\ 2)(1\ 2\ 3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix} = (2\ 3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (1\ 3\ 2)(1\ 2)$$

Any product involving the new elements $(1\ 3)$ and $(2\ 3)$ is covered by the multiplication table of $(1 2 3)$, $(1 3 2)$ and $(1 2)$ which we already established.
Note: With $(1 3)$ and $(2 3)$ included, the set S given in the answer obviously contains all permutations of 3 elements.

3. Only the short answers were required.

(a) $\{A \in SL_3(\mathbb{Z}) : a_{11} = 1\}$ is not a group.
Explanation: We give a counterexample. The matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & a_{23} \\ 1 & 1 & 2 \end{pmatrix}$$

has determinant one for any choice of $a_{23} \in \mathbb{Z}$. The product of any two such matrices has $a_{11} = 3 \neq 1$.

(b) $\left\{A \in \mathbb{R}^{2\times 2} : A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} = \left\{\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} : b, d \in \mathbb{R}\right\}$ is not a group.
Explanation: When $d = 0$, the matrix is not invertible.

(c) $G := \left\{A \in O_2(\mathbb{Q}) : A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ is a group.
Explanation: The orthogonal matrices are invertible by definition ($AA^T = 1 = A^TA$ implies $A^T = A^{-1}$), and we have

$$G = \left\{A = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix},\ b, d \in \mathbb{Q}\ ,\ d \neq 0\ ,\ AA^T = \begin{pmatrix} 1 + b^2 & bd \\ bd & d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ b & b^2 + d^2 \end{pmatrix} = A^TA\right\}$$
$$= \left\{\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} : d \in \mathbb{Q} \setminus \{0\}\right\}\ .$$

G contains the identity matrix. The diagonal matrices are closed under multiplication, and the condition $a_{11} = 1$ is kept up. The inverse of a diagonal matrix is diagonal again, and $\mathrm{diag}(1, d)^{-1} = \mathrm{diag}(1, d^{-1})$. The latter is an element of G again, since $d \in \mathbb{Q} \setminus \{0\}$.