ex1214-3-sol-2013

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1. (a) The equation says that $y^{-1}x^{-1}$ is inverse to xy. (Right inverse:) Multiplication of both sides by (xy) from the left:

$$e = (xy)(xy)^{-1} \stackrel{!}{=} (xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = xx^{-1} = e \quad \text{o.k.}$$

Left inverse: we multiply by (xy) from the right:

$$e = (xy)^{-1}(xy) \stackrel{!}{=} (y^{-1}x^{-1})(xy) = y^{-1}(x^{-1})x)y = y^{-1}y = e \quad \text{o.k.}$$

(b)

$$x^2 = x \quad \Rightarrow \quad x^{-1}x^2 = e \quad \Rightarrow \quad x = e\,.$$

(c)

$$x^3 = x \quad \Rightarrow \quad x^{-1}x^3 = e \quad \Rightarrow \quad x^2 = e\,.$$

Counter example: In the group $(\{-1, 1\}, \cdot)$, the identity is $e = 1$. Both group elements satisfy $x^2 = 1$, but $-1 \neq e$.

2. Let

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}.$$

a is the permutation that sends $(1, 2, 3, 4, 5) \mapsto (1, 4, 3, 2, 5)$ (i.e., it is the identity on the 1st, 3rd and on the 5th object, and it exchanges the positions 2 and 4).

(a) ab (first b, then a) is the permutation that sends

$$(1, 2, 3, 4, 5) \stackrel{b}{\mapsto} (3, 4, 5, 2, 1) \stackrel{a}{\mapsto} (3, 2, 5, 4, 1)\,.$$

Thus $ab = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$.

a^{-1} is the permutation that undoes the permutation a. That is, it exchanges $2 \leftrightarrow 4$ back:

$$(1, 2, 3, 4, 5) \stackrel{a^{-1}}{\mapsto} (1, 4, 3, 2, 5)\,.$$

Thus $a^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix} = a$.

Indeed,

$$(1, 2, 3, 4, 5) \stackrel{a}{\mapsto} (1, 4, 3, 2, 5) \stackrel{a^{-1}}{\mapsto} (1, 2, 3, 4, 5)\,,$$

so $a^{-1}a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = e$ is the identity. and

$$a^{-1} = \begin{pmatrix} 1 & 4 & 3 & 2 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1' & 2' & 3' & 4' & 5' \\ 1' & 4' & 3' & 2' & 5' \end{pmatrix} = a$$

(rename $2' = 4$ and $4' = 2$).

(b) We have to solve

$$ax = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} = b$$

Now

$$b(1, 2, 3, 4, 5) = (3, 4, 5, 2, 1) = a(3, 2, 5, 4, 1)\ ,$$

so

$$x(1, 2, 3, 4, 5) = (x_1, x_2, x_3, x_4, x_5) = (3, 2, 5, 4, 1)\ .$$

We conclude that $x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$.

(c) As a permutation,

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

Here we have inserted rows (intermediate steps of the permutation) so that each row differs from the respective previous row by one transposition (exchange of two positions) only. The first transposition exchanges 1 and 3 (denoted by (1 3)), and the whole series of transpositions is

$$b = (1\ 5)(2\ 4)(1\ 3)\ .$$

Since the number of permutations is odd, the sign is -1.
Note: While the decomposition is not unique, the sign will always remain the same.

(d)

$$b = (1, 3, 5)(2, 4) = (2, 4)(1, 3, 5)\ .$$

3. (a) $S_2 = \{(1), (1\ 2)\}$ is cyclic. Explanation: $(1\ 2)^2 = (1)$.

(b) $n\mathbb{Z} \subset \mathbb{Z}$ is cyclic. Explanation: $(\mathbb{Z}, +)$ is (infinite) cyclic, and $(n\mathbb{Z}, +)$ is a subgroup generated by single element n, thus cyclic.

(c) $(\mathbb{Q}, +)$ is not cyclic. Explanation: Otherwise there would exist $a, b \in \mathbb{Z}$, $b \neq 0$ such that a/b were a single generator of $\mathbb{Z}$, for instance

$$\frac{a}{2b} = n\frac{a}{b}$$

for some $n \in \mathbb{Z}$, contradiction.

(d) The group of rotations in $\mathbb{R}^2$ is not cyclic. Explanation: There is no generating angle ϕ_0 such that every angle of rotation equals $n\phi_0$ for some $n \in \mathbb{Z}$.

(e) The group of matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \in \mathbb{Z}$ is cyclic. Explanation: For $a, b \in \mathbb{Z}$,

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}$$

So the group of such matrices is cyclic as it is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$