ex1214-2-solution

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1. The identity element is the element e such that $e * n = n * e = n$ for all $n \in \mathbb{Q}$.

   (a) There is identity. Explanation:

   $$e * n = 2en \stackrel{!}{=} n \quad \Leftrightarrow \quad e = \frac{1}{2}, \quad \text{or} \quad n = 0.$$

   $e = \frac{1}{2}$ works for all $n \in \mathbb{Q}$.

   (b) There is an identity. Explanation:

   $$e * n = e + n + 1 \stackrel{!}{=} n \quad \Leftrightarrow \quad e = -1.$$

   (c) There is no identity. Explanation:

   $$e * n = \frac{e - n}{2} \stackrel{!}{=} n \quad \Leftrightarrow \quad e = 3n.$$

   e depends on the element to which it is being applied.

   (d) There is no identity. Explanation:

   $$e * n = 11 \stackrel{!}{=} n \quad \Leftrightarrow \quad n = 11.$$

   Not all numbers in $\mathbb{Q}$ are equal to 11. So the condition cannot be satisfied for all n, irrespective of the choice of e.

2. For $x, y \in \{a, b, c\}$, we have $e \cdot x = x = x \cdot e$, so for instance $(e \cdot x) \cdot y = x \cdot y = e \cdot (x \cdot y)$.

3. We need to check closure w.r.t. the group operation and existence of an inverse (in particular the existence of an identity).

   (a) $(S, \cdot)$ with $S = \{-1, 1\}$ is a group. Explanation: $(\pm 1) \cdot (\pm 1) = 1 \in S$, and $(\pm 1)(\mp 1) = -1 \in S$. So the set $\{-1, 1\}$ is closed under multiplication. Moreover, for either sign, ± 1 equals its own inverse (since 1 is the identity).

   (b) $(S, +)$ with $S = \{-1, 0, 1\}$ is not a group. Explanation: For example, $1 + 1 = 2 \notin S$.

   (c) $(S, \cdot)$ with $S = \mathbb{Z}^*$ is not a group. Explanation: For example, 3 has no inverse in S (1 being the identity).

   (d) $(S, +)$ with $S = 5\mathbb{Z}$ is a group. Explanation: Whenever $x, y \in \mathbb{Z}$, we have

   $$5x + 5y = 5(x + y)$$

   in $\mathbb{Z}$. Since $x + y \in \mathbb{Z}$, we have $5(x + y) \in S$, so S is closed under $+$. The equation also shows that $5x \in S$ has inverse $5y$ with $y = -x$ (the identity being 0).

(e) $(S, \cdot)$ with $S = 5\mathbb{Z}$ is not a group. Explanation: S has no multiplictive identity.

(f) $(S, -)$ with $S = \mathbb{Z}$ is not a group, associativity does not hold.

(g) $(S, \cdot)$ with $S = 2^{\mathbb{Z}}$ is a group. Explanation: For $n, m \in \mathbb{Z}$, we have

$$2^n \cdot 2^m = 2^{n+m}$$

in $\mathbb{Z}$. Since $m + n \in \mathbb{Z}$, $2^{n+m} \in 2^{\mathbb{Z}}$, so S is closed under $\cdot$. The equation also shows that $2^n \in S$ has inverse 2^m with $m = -n$ (the identity being 1).