ex1214-1-solutions

March 15, 2015

1. (a) $A = \{1\}$, $B = \{1, 2\}$.
   Solution: 2,2,0,0
   Explanation: The only maps are $1 \mapsto 1$, and $1 \mapsto 2$. In either case, there is only one element in A, so necessarily $f(a) = f(b) \Rightarrow a = b$ for either of the maps, whence it is injective. In either case, one element in B is not reached, so there is no surjective map. In particular, no map is bijective.

   (b) $A = \{1, 2\}$, $B = \{1, 2\}$.
   Solution: 4,2,2,2
   Explanation: The permutations of the ordered pair of numbers $(1, 2)$ give rise to two maps. These maps are both surjective and injective, thus bijective. In addition, there are two constant maps, which send A to 1 and 2, respectively. In either case, one element of B is not reached, so the maps are not surjective. Also, two different elements of A are sent to the same value in B, so they are not injective either. Thus they're not bijective.

   (c) $A = \{1, 2\}$, $B = \{1, 2, 3\}$.
   Solution: 9,6,0,0
   Explanation: Any map is defined by assigning one value to each element in A. For either of the two elements in A, there are 3 possible values. So there are nine maps overall. If I do not allow the two elements of A to take the same value, then I have the choice between 3 values for the first element in A, but only 2 for the second. Thus there are 6 non-constant, injective, maps. The 3 constant maps are not injective. Since A has less elements than B, there is always an element in B that is not reached. Thus there is no surjective map.

2. Inverse maps f^{-1}:

   (a) $f(x) = -5x$
   Solution: $f^{-1}(x) = -\frac{1}{5}x$.

   (b) $f(x) = x + 2$
   Solution: $f^{-1}(x) = x - 2$

   (c) $f(x) = e^x$
   Solution: $f^{-1}(x) = \ln x$.

   Check that the composition yields the identity map, $f^{-1} \circ f = \mathrm{id}$.

3. Let $f \colon S \to T$ be a map and $A, B \subset S$ be two subsets.

   (a) $f(A \cup B) = f(A) \cup f(B)$:
   $\Rightarrow$: If $x \in A \cup B$, then $x \in A$, or $x \in B$. Thus $f(x) \in f(A)$, or $f(x) \in f(B)$,

i.e. $f(x) \in f(A) \cup f(B)$.
$\Leftarrow$: If $y \in f(A) \cup f(B)$, then $y \in f(A)$, or $y \in f(B)$. Thus $\exists\, x$ such that $f(x) = y$ and $x \in A$ or $x \in B$, i.e. $x \in A \cup B$.

(b) $f(A \cap B) \subset f(A) \cap f(B)$
If $y \in f(A \cap B)$, then $\exists\, x \in A \cap B$ such that $y = f(x)$. Since $x \in A$ and $x \in B$, we have $y \in f(A)$ and $y \in f(B)$, i.e. $y \in f(A) \cap f(B)$.
"$\subset$" cannot be replaced by "$=$":
For example, take $A = \{1, -1, -2\}$, $B = \{1, 2, 3\}$, and $f(x) = |x|$.

4. Which binary operations $*$ on the natural numbers $\mathbb{N}$ are commutative and which are associative:

(a) $m * n = mn + 1$: commutative, not associative
Explanation: The product $m * n$ is commutative because this is true for the ordinary product mn between natural numbers. Associativity: For $m \neq u$,

$$(m * n) * u = (mn + 1) * u = (mn + 1)u + 1 = mnu + u + 1$$
$$\neq m * (n * u) = m * (nu + 1) = m(nu + 1) + 1 = mnu + m + 1$$

(b) $m * n = \frac{m-n}{2}$: not commutative, not associative

(c) $m * n = 55$: both commutative and associative