1. (a) $A = \{1\}, B = \{1, 2\}$.
   Solution: 2, 2, 0, 0
   Explanation: The only maps are $1 \mapsto 1$, and $1 \mapsto 2$. In either case, there is only one element in $A$, so necessarily $f(a) = f(b) \Rightarrow a = b$ for either of the maps, whence it is injective. In either case, one element in $B$ is not reached, so there is no surjective map. In particular, no map is bijective.

(b) $A = \{1, 2\}, B = \{1, 2\}$.
   Solution: 4, 2, 2, 2
   Explanation: The permutations of the ordered pair of numbers $(1, 2)$ give rise to two maps. These maps are both surjective and injective, thus bijective. In addition, there are two constant maps, which send $A$ to 1 and 2, respectively. In either case, one element of $B$ is not reached, so the maps are not surjective. Also, two different elements of $A$ are sent to the same value in $B$, so they are not injective either. Thus they’re not bijective.

(c) $A = \{1, 2\}, B = \{1, 2, 3\}$.
   Solution: 9, 6, 0, 0
   Explanation: Any map is defined by assigning one value to each element in $A$. For either of the two elements in $A$, there are 3 possible values. So there are nine maps overall. If I do not allow the two elements of $A$ to take the same value, then I have the choice between 3 values for the first element in $A$, but only 2 for the second. Thus there are 6 non-constant, injective, maps. The 3 constant maps are not injective. Since $A$ has less elements than $B$, there is always an element in $B$ that is not reached. Thus there is no surjective map.

2. Inverse maps $f^{-1}$:

(a) $f(x) = -5x$
   Solution: $f^{-1}(x) = -\frac{1}{5}x$.

(b) $f(x) = x + 2$
   Solution: $f^{-1}(x) = x - 2$

(c) $f(x) = e^x$
   Solution: $f^{-1}(x) = \ln x$.

Check that the composition yields the identity map, $f^{-1} \circ f = \text{id}$.

3. Let $f : S \to T$ be a map and $A, B \subset S$ be two subsets.

(a) $f(A \cup B) = f(A) \cup f(B)$.
   \[\Rightarrow: \text{If } x \in A \cup B, \text{ then } x \in A, \text{ or } x \in B. \text{ Thus } f(x) \in f(A), \text{ or } f(x) \in f(B),\]
i.e. \( f(x) \in f(A) \cup f(B) \).
\[ \Leftarrow: \] If \( y \in f(A) \cup f(B) \), then \( y \in f(A) \), or \( y \in f(B) \). Thus \( \exists x \) such that \( f(x) = y \) and \( x \in A \) or \( x \in B \), i.e. \( x \in A \cup B \).

(b) \( f(A \cap B) \subset f(A) \cap f(B) \)
If \( y \in f(A \cap B) \), then \( \exists x \in A \cap B \) such that \( y = f(x) \). Since \( x \in A \) and \( x \in B \), we have \( y \in f(A) \) and \( y \in f(B) \), i.e. \( y \in f(A) \cap f(B) \).

“\( \subset \)” cannot be replaced by “\( = \)”:
For example, take \( A = \{1, -1, -2\} \), \( B = \{1, 2, 3\} \), and \( f(x) = |x| \).

4. Which binary operations \( * \) on the natural numbers \( \mathbb{N} \) are commutative and which are associative:

(a) \( m * n = mn + 1 \): commutative, not associative

**Explanation:** The product \( m \cdot n \) is commutative because this is true for the ordinary product \( mn \) between natural numbers. Associativity: For \( m \neq u \),

\[
(m * n) * u = (mn + 1) * u = (mn + 1)u + 1 = mnu + u + 1
\]

\[
\neq m * (n * u) = m * (nu + 1) = m(nu + 1) + 1 = mnu + m + 1
\]

(b) \( m * n = \frac{m + n}{2} \): not commutative, not associative

(c) \( m * n = 55 \): both commutative and associative