October 21, 2018 Lecturer Dmitri Zaitsev

Course 1213 - Introduction to group theory 2018

S h e e t 6

Due: at the end of the tutorial

Exercise 1

Which sets of matrices form a group under multiplication:

(i) $\left\{A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL_2(\mathbb{Z}) : a_{11}a_{22} = \pm 1\right\}$;

(ii) $\left\{A \in GL_2(\mathbb{R}) : A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$;

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(ii) $\left\{A \in SO_2(\mathbb{Q}) : A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$;

(iii) $\left\{A \in O_2(\mathbb{Q}) : A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$;

(iv) $\left\{A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in U_2 : a_{11} = 1\right\}$.

Exercise 2

Prove or disprove:

(i) If $a|b$ (a divides b) and $b|c$, then $a|c$.

(ii) If $a|bc$, then $a|b$ or $a|c$.

(iii) If $a|b$ and $b|a$, then $a = \pm b$.

(iv) If $a|b$, then $a^2|b^2$.

(iv) If $a|b$, then $(a^2 + a)|(b^2 + b)$.

Exercise 3

Use the Euclidean algorithm to compute the greatest common divisor:

(i) $\gcd(1045, 33)$

(ii) $\gcd(56, 182)$

(iii) $\gcd(234, 2575)$.

Express each greatest common divisor as integer linear combination of the two given integers.

Exercise 4

Use the unique prime factorization to prove:

(i) $\gcd(ac, bc) = c \gcd(a, b)$ for all integers a, b, c.

(ii) If $\gcd(a, c) = \gcd(b, c) = 1$, then $\gcd(ab, c) = 1$.