MA5P8 CONFORMAL FIELD THEORY

Sections 4 and 5: Defining CFTs

1. Prove that for a system $\langle \cdots \rangle$ of n-point functions on $\mathbb{H}$ with pairwise quasi-primary $\varphi_i \in \mathbb{H}_{h_i,\overline{h}_i}$,

$$\begin{aligned}
\langle \varphi_i(z_i) \rangle &= C_i \in \mathbb{C}, \\
\langle \varphi_1(z_1)\varphi_2(z_2) \rangle &= \frac{C_{12}}{z_{12}^{h_1+h_2}\overline{z}_{12}^{\overline{h}_1+\overline{h}_2}}, \\
\langle \varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_3) \rangle &= \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{31}^{h_3+h_1-h_2} \overline{z}_{12}^{\overline{h}_1+\overline{h}_2-\overline{h}_3} \overline{z}_{23}^{\overline{h}_2+\overline{h}_3-\overline{h}_1} \overline{z}_{31}^{\overline{h}_3+\overline{h}_1-\overline{h}_2}}
\end{aligned}$$

   with $C_i = 0$ unless $\varphi_i \in \mathbb{H}_{0,0}$, $C_{12} = 0$ unless $h_1 = h_2$ and $\overline{h}_1 = \overline{h}_2$, and $C_{123} \in \mathbb{C}$.

2. Show that in a conformal representation of an OPE where reflection positivity holds and the vacuum is normalised to $OPE(\Omega \otimes Y) = Y$ for all $Y \in \mathbb{H}$, $T = L_2\Omega \in \mathbb{H}_{2,0}$ obeys

$$\begin{aligned}
\lim_{z,w\to 0} \langle T^\dagger(\overline{z}^{-1}) T(w) \rangle &= \tfrac{c}{2} = \langle T, T \rangle, \\
\lim_{z,w\to 0} \langle T^\dagger(\overline{z}^{-1}) \partial T(w) \rangle &= 0, \\
\lim_{z,w\to 0} \langle (\partial T)^\dagger(\overline{z}^{-1}) \partial T(w) \rangle &= 2c = \langle L_1 T, L_1 T \rangle.
\end{aligned}$$