MA5P8 CONFORMAL FIELD THEORY

Section 2: The free fermion (Ising) model

1. Remind yourself of the definitions of the Fock space representation of the free fermion algebra (FFF) and the Virasoro modes in terms of the FFF (carefully distinguishing two sectors).

2. Calculate $[L_n, L_m]$ for $m+n = 0$ and $[L_n, \psi_k]$ in the FFF, using the definitions given in the previous question.

3. Show that if $[L_n, L_m]\Psi = \left\{(m-n)L_{m+n} + \frac{c}{12}\delta_{m+n,0}m(m^2-1)\right\}\Psi$ for a state Ψ in the Fock space then also $[L_n, L_m]\psi_k\Psi = \{(m-n)L_{m+n} + \frac{c}{12}\delta_{m+n,0}m(m^2 -1)\}\,\psi_k\Psi$ for all ψ_k.

4. Consider a unitary lwr of $\mathcal{V}\mathrm{ir}_{\frac{1}{2}}$ with lwv v of weight $\frac{1}{2}$ and determine $a,\, b \in \mathbb{R}$ such that $aL_2 v + bL_1^2 v = 0$.

5. Show that
$$P(w,q) := \prod_{n=1}^{\infty}\left(1 + w\, q^{n-\frac{1}{2}}\right)\left(1 + w^{-1}\, q^{n-\frac{1}{2}}\right) = \left[\prod_{n=1}^{\infty}(1-q^n)\right]^{-1}\sum_{k=-\infty}^{\infty} w^k\, q^{\frac{k^2}{2}},$$
in the region where P is well defined.

   [Hint: A sketch of the proof was given in the lectures.]

6. With the Jacobi theta functions ϑ_n,
$$\vartheta_2(q) = \sum_{k=-\infty}^{\infty} q^{\frac{1}{2}(k+\frac{1}{2})^2}, \quad \vartheta_3(q) = \sum_{k=-\infty}^{\infty} q^{\frac{1}{2}k^2}, \quad \vartheta_4(q) = \sum_{k=-\infty}^{\infty} (-1)^k\, q^{\frac{1}{2}k^2},$$
and the Dedekind eta function η, show that
$$\frac{1}{2}\left(\left|\frac{\vartheta_2}{\eta}\right|^2 + \left|\frac{\vartheta_3}{\eta}\right|^2 + \left|\frac{\vartheta_4}{\eta}\right|^2\right) = \frac{1}{|\eta|^2}\sum\nolimits_{n,l} q^{\frac{1}{4}(\frac{n}{\sqrt{2}}+l\sqrt{2})^2}\overline{q}^{\frac{1}{4}(\frac{n}{\sqrt{2}}-l\sqrt{2})^2}.$$

7. Show that the Fourier transform $\widetilde{f}$ (where $\widetilde{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi i x y} f(x)\,dx$) of $f(x) := e^{-\left(\sqrt{\pi a}\frac{x}{r} - \frac{b}{2\sqrt{\pi a}}\right)^2}$ is $\widetilde{f}(y) = \frac{r}{\sqrt{a}}e^{-\frac{y^2 r^2 \pi}{a} - \frac{iybr}{a}}$.

8. Using Poisson resummation (see your notes) and the previous question, show that, provided both sides converge absolutely, for $a,\, b \in \mathbb{C}$
$$\sum_{n=-\infty}^{\infty} e^{-\pi a n^2 + bn} = \frac{1}{\sqrt{a}}\sum_{k=-\infty}^{\infty} e^{-\frac{\pi}{a}\left(k - \frac{b}{2\pi i}\right)^2}.$$

9. For $q = \exp(2\pi i\tau)$ and $\widetilde{q} = \exp(-2\pi i/\tau)$, using the previous question, calculate $\vartheta_k(\widetilde{q})$ in terms of $\vartheta_k(q)$ for the Jacobi theta functions $\vartheta_2,\, \vartheta_3,\, \vartheta_4$.