Monstrous Moonshine

A lecture at Albert Ludwigs Universität Freiburg

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0 Introduction

In 1979, J McKay noticed the following equality: *16 Oct 2017*

$$\underbrace{196884}_{\substack{q^1 \text{ coefficient of} \\ \text{the modular ``}\jmath\text{-function''}}} = \underbrace{196883 + 1}_{\substack{\text{dimensions of the smallest} \\ \text{irreducible representations} \\ \text{of the Monster } \mathbb{M}}}$$

The obvious question now is:

How and why can this be?

Our lecture evolutes around a partial answer:

Conformal Field Theory connects the two.

Let us first look at the right-hand side of the equation above: In 1983, Gorenstein announced the classification of simple finite groups. We shall briefly discuss the following

Definition 0.1. A group G is simple iff

$$\forall H \lhd G : H \in \{G, \{1\}\}$$

As an exercise, it can be shown, that $H \lhd G$ iff ${}^{G}/_{H}$ forms a group with the induced composition. Finite simple groups are the smallest building blocks of finite groups. The classification mentioned above however was just completed by Aschbacher in 2004 and is built on more than 200 publications of various authors. In it a nutshell, the statement is the following

Theorem. *If G is a finite simple group, then G belongs either to a list of 18 infinite families (...), or is one of 26 so-called sporadic groups.*

Now $\mathbb{M}$, the Monster-Group, denotes the biggest of the sporadic groups. Its order is approximately $8 \cdot 10^{53}$.
If we now consider a $\mathbb{C}$-vectorspace V such that a homomorphism $\mathbb{M} \hookrightarrow \mathrm{GL}(V)$ exists, $\dim V \geq 196883$ holds, with the given bound being exact.

Let us now have a look at the left-hand side of the initial equation.

Definition 0.2. Let $\mathfrak{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ denote the upper half plane and $\overline{\mathfrak{H}}$ its union with the cusps $\mathbb{Q} \cup \{\imath\infty\}$.

(a) $\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ let $\mathfrak{H} \to \mathfrak{H} : \tau \mapsto A.\tau := \frac{a\tau+b}{c\tau+d}$.
This map is called a Möbius transform.

(b) A group $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is called commensurable with $\mathrm{SL}_2(\mathbb{Z})$ iff $\Gamma \cap \mathrm{SL}_2(\mathbb{Z})$ has finite index[1] in both Γ and $\mathrm{SL}_2(\mathbb{Z})$.

Remark 0.3.

- $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathfrak{H}$, i.e. there is a natural group- homomorphism

$$\mathrm{SL}_2(\mathbb{R}) \to \{f : \mathfrak{H} \hookrightarrow\!\!\!\!\rightarrow \mathfrak{H}\}$$

, so $\mathrm{SL}_2(\mathbb{Z})$ and all groups that are commensurable with $\mathrm{SL}_2(\mathbb{Z})$ also act.

[1] Let G, H be groups, $G \supset H$. Define the index of H in G as $|G : H| := \#({}^{G}/_{H})$

- For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the action defined above naturally extends to $\overline{\mathfrak{H}}$ and the cusps form a single orbit. One can even show the above for any Γ commensurable with $\mathrm{SL}_2(\mathbb{Z})$. It can be shown, that the quotient[2][3]

$$\Gamma\backslash\overline{\mathfrak{H}} := \overline{\mathfrak{H}}/\sim_\Gamma \quad \text{where} \quad z \sim_\Gamma z' :\Longleftrightarrow \exists\, \gamma \in \Gamma : \gamma.z = z'$$

is a compact[4] Riemann surface which can be classified by genus[5]. For example

$$\mathrm{SL}_2(\mathbb{Z})\backslash\overline{\mathfrak{H}} \simeq S^2$$

Definition 0.4. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be commensurable with $\mathrm{SL}_2(\mathbb{Z})$.

(a) A function $f : \overline{\mathfrak{H}} \to \overline{\mathbb{C}}$ is called MODULAR FUNCTION WITH RESPECT TO Γ if the following statements hold:

- f is MEROMORPHIC, i.e. for any $\tau_0 \in \mathfrak{H}$ there is a neighborhood U and $(a_k)_{k\geq -N}$, such that

$$\forall\, \tau \in U : f(\tau) = \sum_{k=-N}^{\infty} a_k(\tau - \tau_0)^k$$

  Actually some appropriate behavior at the cusps is required, but we delegate the details for now to a later point.

- $$\forall\, \tau \in \overline{\mathfrak{H}} : \forall\, A \in \Gamma : f(A.\tau) = f(\tau)$$

  i.e. f induces a function $\overline{f} : {}_\Gamma\backslash\overline{\mathfrak{H}} \to \overline{\mathbb{C}}$ in a natural way.

(b) If ${}_\Gamma\backslash\overline{\mathfrak{H}}$ has genus zero, then Γ is called a "GENUS ZERO GROUP". If additionally Γ contains only integer shifts, i.e. if

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma \overset{!}{\Longrightarrow} b \in \mathbb{Z}$$

holds, then a modular function f with respect to Γ is called HAUPTMODUL if $\overline{f}$ is bijective and if there is a sequence $(a_n)_{n\geq 1}$ such that

$$\forall\, \tau \in \overline{\mathfrak{H}} : f(\tau) = q^{-1} + \sum_{n=1}^{\infty} a_n q^n \quad \text{where} \quad q = \exp(2\pi\imath\tau)$$

Remark 0.5.

- For any Γ as in Definition 0.4 (b), there is an unique Hauptmodul.
- The MODULAR j-FUNCTION is defined as $j(\tau) = J(\tau) + 744$ where J is the Hauptmodul for $\mathrm{SL}_2(\mathbb{Z})$. It can be expressed as

$$j(\tau) = \frac{\left(1 + \sum\limits_{n=1}^{\infty} \sigma_3(n) q^n\right)^3}{q \cdot \prod\limits_{n=1}^{\infty} (1 - q^n)^{24}} \quad \text{where} \quad \sigma_3(n) = \sum_{d|n} d^3$$

[2] Γ is written on the left side of the quotient, since it acts from the left.

[3] The classes are called ORBITS.

[4] 'Compact' means closed and compact in this context. This is a misuse of terminology commonly committed in differential geometry.

[5] The GENUS (german Geschlecht) of a compact, closed, orientable surface F can be thought of as its number of holes, so S^2 has genus zero, a torus has genus one, etc.
It is formally defined as the maximum possible number of cuts (where cuts are exclusions of nonintersecting simple closed curves), that cannot introduce a new connected component.

We can now finally state the

Theorem 0.6. (Moonshine Conjectures) [Conway, Norton 79]

(i) *There is a infinite dimensional* $\mathbb{C}$*-vectorspace* $V^\natural = V_0 \oplus V_1 \oplus V_2 \oplus \ldots$*, such that*

$$J(\tau) = q^{-1} \cdot \sum_{n=0}^{\infty} (\dim V_n) q^n \qquad \textit{i.e.} \quad V_0 \simeq \mathbb{C},\ V_1 \simeq \{0\}\,,\ \dim V_2 = 192883\ldots$$

and for any $n > 1$ *one can find* $\rho_n : \mathbb{M} \hookrightarrow \mathrm{GL}(V_n)$*, where* ρ_n *is a group homomorphism*[6].

(ii) *For any* $g \in \mathbb{M}$ *define the* McKay Thompson series *as in* [Thompson 79]

$$T_g(\tau) := q^{-1} \cdot \sum_{n=1}^{\infty} \mathrm{tr}(\rho_n(g))\, q^n$$

Then for any $g \in \mathbb{M}$ *there exists a subgroup* $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$*, such that* T_g *obeys the "*genus zero condition*" with respect to* Γ_g*, i.e.* Γ_g *is commensurable with* $\mathrm{SL}_2(\mathbb{Z})$*, has genus zero, and* T_g *is the Hauptmodul for* Γ_g*.*

Remark 0.7.

- For $g = \mathrm{id}$, $\mathrm{tr}(\rho_n(g)) = \dim V_n$ is apparent. Thus by part (i) of the Moonshine Conjectures, $T_{\mathrm{id}} = J$ should hold.
- There are 616 genus zero groups, only 171 of which appear as Γ_g in Monstrous Moonshine. All of the latter obey

$$\Gamma_g \supset \Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod N \right\}$$

  for some N with $N|24$ and $N|\,\mathrm{ord}(g)$.
- The word Moonshine refers both to a lunatic (i.e. Conway) and to illegally brewed alcohol (in the US during the prohibition). The latter connotation also applies to Monstrous Moonshine in a certain way, since we somehow "illegally" obtain and smuggle information from one side of the theory to the other.
- The Moonshine Conjectures are true. (!)

The history of Monstrous Moonshine can be summarized as follows:

1965: Leech[7] constructs the Leech lattice, which talks to both sides of Monstrous Moonshine.

1973: Fischer[8] and Griess[9] independently conjecture the existence of $\mathbb{M}$.

1979: Thompson[10] introduces T_g and Conway[11] and Norton[12] state the Moonshine Conjectures.

1980: Griess proves the existence of $\mathbb{M}$.

[6] In other words: $V^\natural$ carries an infinite dimensional graded representation of $\mathbb{M}$
[7] John Leech (1926 - 1992)
[8] Bernd Fischer (born 1926)
[9] Robert Griess (born 1946)
[10] John Griggs Thompson (born 1932), Fields Medal 1970, Abel Prize 2008
[11] John Horton Conway (born 1937), invented Game Of Life
[12] Simon Phillips Norton (born 1952), contributed to Game Of Life

1984: Frenkel[13], Lepowsky[14] and Meurman[15] construct $V^\natural$ and $(\rho_n)_{n>1}$ using methods from Quantum Field Theory.

1992: Borcherds[16] develops the theory of vertex operator algebras (VOAs) entering in Conformal Field Theory (CFT), alongside the theory of certain infinite dimensional Lie algebras (Borcherds-Kac[17]-Moody[18]) obtained from VOAs.

1998: Fields Medal for Borcherds

2017: Carnahan[19] fills the last remaining gaps in the proof.

18 Oct 2017 **Remark 0.8.** (Structure of the Proof)

(1) Construction of $V^\natural$ by methods from CFT (VOAs), starting from the Leech lattice.

(2) Construction of the Monster Algebra $\mathfrak{m}$, an infinite-dimensional Lie algebra of Borcherds-Kac-Moody type, thus naturally generalizing the (complex, finite-dimensional) semisimple Lie algebras.

(3) Structure theory of $\mathfrak{m}$: allows to relate the behavior of $V^\natural$ under $\mathbb{M}$ to the modular properties of T_g.

(4) Obtain recursion formula for the coefficients of the T_g, which agree with the recursive relations for the Hauptmoduln of Γ_g and then complete the proof by comparing the first few coefficients.

The proof outlined above is the goal of this lecture and its structure is determined by the ingredients of the proof:

§1 Finite Groups, $\mathbb{M}$ in particular and its connection to the Leech lattice

§2 Lie algebras

§3 Modular Forms and their relation to Lie algebras

§4 Vertex Operator Algebras

§5 The proof

[13] Igor Borisovich Frenkel (born 1952), Russian National, PhD 1980 in Yale
[14] John Lepowski (born 1944)
[15] Arne Meurman (born 1956)
[16] Richard Borcherds (born 1959), student of Conways'
[17] Victor Gershevich Kac (born 1943)
[18] Robert Vaughan Moody (born 1941)
[19] Scott Carnahan, PhD student of Borcherds'

1 Groups, Lattices and the Monster

Definition 1.1. Let $N, \tilde{G}, G$ denote groups, such that a SHORT EXACT SEQUENCE[1] exists:

$$1 \to N \overset{\iota}{\hookrightarrow} \tilde{G} \overset{\pi}{\twoheadrightarrow} G \to 1$$

(1) We say that $\tilde{G}$ is an EXTENSION OF G BY N.

(2) If there is a group homomorphism $t : G \to \tilde{G}$ such that $\pi \circ t = \mathrm{id}_G$, we say that the sequence SPLITS and $\tilde{G}$ is called a SEMIDIRECT PRODUCT of G and N.

Remark 1.2. Let $N, \tilde{G}, G$ denote groups as in Definition 1.1 and let $\tilde{N} := \iota(N) \subset \tilde{G}$. As an exercise, the following statements can be shown:

- $\tilde{N} \lhd \tilde{G}$ and ${}^{\tilde{G}}/_{\tilde{N}} \simeq G$
- If $N \simeq \mathbb{Z}_2$, then $\tilde{N} \subset \tilde{G}$ is central, i.e. $\forall\, n \in \tilde{N}, g \in \tilde{G} : ng = gn$.
- If $t : G \to \tilde{G}$ is a splitting, then let

$$\Phi : N \times G \to \tilde{G} : (n, g) \mapsto \iota(n) \cdot t(g)$$

We claim that Φ is bijective. For the proof, $g = \pi(\Phi(n, g))$ is a useful hint.
Note that for any $n, n' \in N, g, g' \in G$

$$\begin{aligned}\Phi(n, g) \cdot \Phi(n', g') &= \iota(n) \cdot t(g) \cdot \iota(n') \cdot t(g') \\ &= \iota(n) \cdot \underbrace{t(g) \cdot \iota(n') \cdot (t(g))^{-1}}_{=:\iota(\alpha_g(n'))} \cdot \underbrace{t(g) \cdot t(g')}_{t(gg')}\end{aligned}$$

Our claim is, that for any $g \in G$, $\alpha_g : N \to N$ is well defined in the equation above:

$$\begin{aligned}&\exists!\, \alpha_g(n) \in N : \iota(\alpha_g(n)) = t(g) \cdot \iota(n) \cdot (t(g))^{-1} \\ \implies\ &\Phi(n, g) \cdot \Phi(n', g') = \iota(n\, \alpha_g(n')) \cdot t(gg') = \Phi(n\, \alpha_g(n'), gg')\end{aligned}$$

- For any $g \in G$, α_g is an automorphism on N. Further, the map $\alpha : g \mapsto \alpha_g$ is a homomorphism:

$$\forall\, g, g' \in G : \alpha_g \circ \alpha_{g'} = \alpha_{gg'}$$

- Let $\alpha \in \mathrm{Hom}(G, \mathrm{Aut}(N))$. On the set $N \times G$ a group structure can be defined with the following operation:

$$(n, g) \cdot (n', g') = (n\, \alpha_g(n'), gg')$$

The standard notation is $N \rtimes_\alpha G = G \ltimes_\alpha N := (N \times G, \cdot\,)$

Given G, N as above, our objective is now to construct all possible groups $\tilde{G}$ such that $N \lhd \tilde{G}$, $G = {}^{\tilde{G}}/_{N}$. This behavior can be expressed neatly with a short exact sequence *21 Oct 2017*

$$1 \to N \overset{\iota}{\hookrightarrow} \tilde{G} \overset{\pi}{\twoheadrightarrow} G \to 1$$

Semidirect products do the job for the special case that a splitting exists. However, we will see that the general case cannot be handled as a semidirect product.

[1] A sequence is exact, if it is exact at every node, meaning that the image of the incoming and the kernel of the outgoing morphism agree. The exactness of the sequence above implies the injectivity of ι and the surjectivity of π.

Example 1.3.

1.) Let $n \in \mathbb{N}, N = \mathbb{R}^n, G = \mathrm{GL}_n(\mathbb{R})$. Define $\alpha \in \mathrm{Hom}(G, \mathrm{Aut}(N))$ as $\alpha_A(b) = Ab$ for any $A \in N$. Apparently, there is a group action:

$$\mathbb{R}^n \rtimes_\alpha \mathrm{GL}_n(\mathbb{R}) \hookrightarrow \mathrm{Bij}(\mathbb{R}^n, \mathbb{R}^n) \quad \text{where} \quad (b, A) \,.\, x = b + Ax$$

Thus, $\mathbb{R}^n \rtimes_\alpha \mathrm{GL}_n(\mathbb{R}) \simeq \mathrm{Aff}(\mathbb{R}^n)$ is the affine linear group. (exercise!)

2.) Let $\tilde{G} := \{1, -1, \imath, -\imath\} \simeq \mathbb{Z}_4$ and $N := \{1, -1\} \simeq \mathbb{Z}_2 \lhd \tilde{G}$. Consider the following quotient:

$$G := \tilde{G} \big/ N \simeq \mathbb{Z}_2$$

If we write down the corresponding short exact sequence

$$1 \to \mathbb{Z}_2 \overset{\iota}{\hookrightarrow} \mathbb{Z}_4 \overset{\pi}{\twoheadrightarrow} \mathbb{Z}_2 \to 1,$$

we note there cannot be a splitting. Indeed, given any $t \in \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_4), t(\pi(1)) = 1$ holds, while $t(-1)^2 = t\big((-1)^2\big) = t(1) = 1$. Since $-1 \in \iota(\mathbb{Z}_2) = \ker \pi$, we can write

$$\pi(t(-1)) = \pi(\pm 1) = 1 \neq -1$$

Thus, t is not a splitting.

The upshot so far is: *Semidirect products do not suffice.*
We shall now look for an appropriate generalization of this concept, so non-splitting short exact sequences are also covered.

Proposition 1.4. *Consider an extension $\tilde{G}$ of a group G by a group N given by the following short exact sequence:*

$$1 \to N \overset{\iota}{\hookrightarrow} \tilde{G} \overset{\pi}{\twoheadrightarrow} G \to 1$$

Choose a map $t : G \to \tilde{G}$ such that $\pi \circ t = \mathrm{id}_G$ and[2] *$t(1) = 1$. Then the following holds:*

(1) For any $\tilde{g} \in \tilde{G}$, there is a unique $(n, g) \in N \times G$ such that $\tilde{g} = \iota(n) \cdot t(g)$.

(2) For any $g \in G$, a unique $\alpha_g \in \mathrm{Aut}(N)$ exists such that

$$\iota(\alpha_g(n)) = t(g) \cdot \iota(n) \cdot (t(g))^{-1} \quad \forall\, n \in N$$

(3) Given $g, g' \in G$, there is exactly one $\nu(g, g') \in N$ such that[3]

$$t(g) \cdot t(g') = \iota(\nu(g, g')) \cdot t(gg').$$

(4) Let α, ν as above. Then for any $g, g' \in G$

$$\alpha_g \circ \alpha_{g'} = C_{\nu(g,g')} \circ \alpha_{gg'} \quad \text{where} \quad C_n(n') = n\, n'\, n^{-1}$$

(5) Considering $g_1, g_2, g_3 \in G$,

$$\nu(g_1, g_2) \cdot \nu(g_1 g_2, g_3) = \alpha_{g_1}(\nu(g_2, g_3)) \cdot \nu(g_1, g_2 g_3)$$

holds[4].

[2] Actually the latter condition is contingent, it just simplifies some expressions.
[3] In a certain way, ν 'measures' how much t is not a homomorphism.
[4] If N is abelian, this is called "cocycle condition" and any ν satisfying this condition is called cocycle.

Proof. (1) and (2) are shown as in Remark 1.2.

(3) With all ingredients as in the theorem, we can write

$$\pi\Big(t(g)\cdot t(g')\cdot (t(gg'))^{-1}\Big) = gg'(gg')^{-1} = 1$$

Since $\ker\pi = \operatorname{im}\iota$, we have just shown

$$\exists\, n\in N : \iota(n) = t(g)\cdot t(g')\cdot (t(gg'))^{-1}$$

Uniqueness follows from the injectivity of ι.

(4) Let $n\in N$ be arbitrary. Then

$$\begin{aligned}\iota(\alpha_g\circ\alpha_{g'}(n)) &\overset{(2)}{=} t(g)\cdot t(g')\cdot t(g')\cdot\iota(n)\cdot (t(g'))^{-1}\cdot (t(g))^{-1}\\ &\overset{(3)}{=} \iota(\nu(g,g'))\cdot\underbrace{t(gg')\cdot\iota(n)\cdot (t(gg'))^{-1}}_{\iota(\alpha_{gg'}(n))\text{ by (2)}}\cdot(\iota(\nu(g,g')))^{-1}\\ &= \iota(C_{\nu(g,g')}\circ\alpha_{gg'}(n))\end{aligned}$$

The injectivity of ι now implies

$$\alpha_g\circ\alpha_{g'}(n) = C_{\nu(g,g')}\circ\alpha_{gg'}(n)$$

as claimed.

(5) This is apparent with $(t(g_1)\cdot t(g_2))\cdot t(g_3) = t(g_1)\cdot (t(g_2)\cdot t(g_3))$ as well as (3) and (4).

□

Proposition 1.5. *If N,G are groups and $\alpha\in\operatorname{Map}(G,\operatorname{Aut}(N))$ $\nu\in\operatorname{Map}(G\times G,N)$ such that (4) and (5) in Proposition 1.4 hold and, in addition*[5]*, $\nu(g,1)=\nu(1,g)=1$ for any $g\in G$ and $\alpha_1=\operatorname{id}_N$, then the operation defined as*

$$(n,g)\cdot(n',g') := (n\,\alpha_g(n')\nu(g,g'), gg')$$

induces the structure of a group on $N\times G$, yielding an extension of G by N.

Proof. Exercise. □

The special case that ν trivial implies $t\in\operatorname{Hom}(G,N\times G)$ and $\alpha\in\operatorname{Hom}(G,\operatorname{Aut}(N))$, giving the semidirect product. Thus we succeeded in generalizing semidirect products.

Sidenote. [6] The proof just conferred to the reader is technical. If the additional condition in the theorem is dropped (as was done in the tutorial group), it becomes a good exercise for keeping track of *lots* of symbols. Thus, I just summarize the details I found worth remembering:

- The identity element in $N\times G$ is given with $(\nu(1,1)^{-1},1)$
- Inverse elements can be written as

$$(n,g)^{-1} = \Big(\nu(1,1)\cdot\big(\nu(g^{-1},g)\big)^{-1}\cdot\big(\alpha_{g^{-1}}(n)\big)^{-1},\ g^{-1}\Big)$$

[5] Again, this latter condition is actually non-essential, but as an additional requirement, it helps simplify things. It is related to the choice $t(1)=1$ in Proposition 1.4.

[6] Sidenotes are *not* an official part of the lecture, but deliberate additions of the typist, trying to incorporate key lessons learned from the exercises or stated verbally in the lecture.

- The maps defined as

$$\iota : N \to N \times N : n \mapsto \Big((\nu(1,1))^{-1}, 1 \Big)$$
$$\pi : N \times G \to G \ : (n, g) \mapsto g$$

are group-homomorphisms, such that

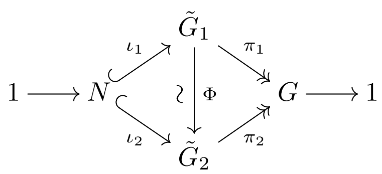
$$1 \to N \overset{\iota}{\hookrightarrow} N \times G \overset{\pi}{\twoheadrightarrow} G \to 1$$

is the desired extension.

Definition 1.6. Consider groups N, G.

(1) A pair (α, ν), $\alpha : G \to \mathrm{Aut}(N)$, $\nu : G \times G \to N$ such that Proposition 1.4 (4), (5) hold is called a PARAMETER SYSTEM of G in N

(2) If for $k \in \{1, 2\}$, $1 \to N \overset{\iota_k}{\hookrightarrow} \tilde{G}_k \overset{\pi_k}{\twoheadrightarrow} G \to 1$ are extensions of G by N, then these are called EQUIVALENT iff there is an isomorphism of groups $\Phi : \tilde{G}_1 \to \tilde{G}_2$ such that



commutes.

(3) Two parameter systems (α, ν), (α', ν') of G in N are EQUIVALENT iff there is a map $f : G \to N$ such that for any $g, g' \in G$

$$\alpha'_g = C_{f(g)} \circ \alpha_g$$
$$\text{and} \quad \nu'(g, g') = f(g) \cdot \alpha_g(f(g')) \cdot \nu(g, g') \cdot (f(gg'))^{-1}$$

both hold.

Exercise. The definition of equivalence in Definition 1.6 (3) amounts to the freedom of the choice of $t : G \to \tilde{G}$ with $\pi \circ t = \mathrm{id}_G$, i.e. $t'(g) = \iota(f(g)) \cdot t(g)$. (If $t'(1) = 1 = t(1)$, then $f(1) = 1$.) As a hint, note that using notion of equivalence in Definition 1.6 (3), without loss of generality $\forall\, g \in G : \nu(1, g) = 1 = \nu(g, 1)$, $\alpha_1 = \mathrm{id}_N$ can be assumed.

Theorem 1.7. [Schreier, 1926] *Let G, N be groups, then the constructions in Proposition 1.4, 1.5 and Definition 1.6 yield naturally inverse bijections between*

$$\mathrm{Ext}(G, N) := \text{set of equivalence classes of extensions of } G \text{ by } N$$
$$\textit{and} \quad \mathrm{Par}(G, N) := \text{set of equivalence classes of parameter systems of } G \text{ in } N.$$

Remark 1.8. Fix two groups N and G.

(1) By Schreiers Theorem above, semidirect products $N \rtimes G$ are classified, up to equivalence by

$$\{[(\alpha, 1)] \in \mathrm{Par}(G, N)\} \quad \text{where} \quad 1 : G \times G \to N : (g, g') \mapsto 1$$

One can define H^1_α, such that this classifies all possible choices of splittings. Unless N is abelian, this is only a 'pointed set' and higher cohomology is not defined.

(2) If N is abelian, $\alpha \in \mathrm{Hom}(G, \mathrm{Aut}(N))$ by Proposition 1.4 (4), and (5) is the "cocycle condition for 2-cocycles in group cohomology with values in N". Thus, the 2-cocycles are just

$$Z^2_\alpha := \{\nu : G \times G \to N \mid \text{Proposition 1.4 (5)}\}$$

Considering Definition 1.6 (3), we note that the commutativity of N implies that for equivalent (α, ν) and (α', ν'), $\alpha = \alpha'$ holds. Thus, we can write $\mathrm{Par}(G, N)$ as a disjoint union:

$$\begin{aligned} \mathrm{Par}(G, N) &= \bigsqcup_{\alpha \in \mathrm{Hom}(G, \mathrm{Aut}(n))} \mathrm{Par}_\alpha(G, N) \\ \text{with} \quad \mathrm{Par}_\alpha(G, N) &= \{[(\alpha, \nu)] \in \mathrm{Par}(G, N)\} \\ &= Z^2_\alpha \big/ B^2_\alpha \end{aligned}$$

The coboundaries above are defined as

$$B^2_\alpha(G, N) = \left\{ \nu \in Z^2_\alpha(G, N) \;\middle|\; \begin{array}{c} \exists\, f \in \mathrm{Map}(G, N) \text{ such that} \\ \nu(g, g') = f(g) \cdot f(g) \cdot \alpha_g\Big(f(g') \cdot (f(gg'))^{-1}\Big) \end{array} \right\}$$

Notation 1.9. If $\tilde{G}$ is an extension of G by N, then write $\tilde{G} = N.G$

Example 1.10. If p is prime and $n \in \mathbb{N} \setminus (0)$, then there exist two inequivalent extensions of $(\mathbb{Z}_p)^{2n}$ by $\mathbb{Z}_p$,

$$1 \to \mathbb{Z}_p \overset{\iota}{\hookrightarrow} p^{1+2n} \overset{\pi}{\twoheadrightarrow} (\mathbb{Z}_p)^{2n} \to 1$$

such that $\mathbb{Z}_p$ is the center of the extension[7], i.e.

$$\iota(\mathbb{Z}_p) = Z(p^{1+2n}) := \left\{ h \in p^{1+2n} \;\middle|\; \forall\, \tilde{g} \in p^{1+2n} : \tilde{g}\, h = h\, \tilde{g} \right\}$$

When the distinction between the two inequivalent extension is of importance we denote them as p^{1+2n}_+ and p^{1+2n}_-.

The extensions in the example above are called "extra special groups" and in Moonshine, 2^{1+24}_+ will play a role. Thus, our next objective is to construct 2^{1+24}_+ in a certain way so that it can be conducive to our subject. *23 Oct 2017*

Definition 1.11. Let $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$ denote a real vectorspace of finite dimension n together with a non-degenerate symmetric bilinear form.

(1) A LATTICE in V is a set $\Lambda \subset V$ of the form

$$\Lambda = \left\{ \sum_{i=1}^{r} n_i f_i \;\middle|\; n_i \in \mathbb{Z} \right\}$$

with linearly independent $f_1, \dots, f_r$.
Λ forms a free $\mathbb{Z}$-module with the natural addition and the multiplication by scalars in $\mathbb{Z}$.
$\mathrm{rk}(\Lambda) := r$ is called the RANK of Λ and $(f_1, \dots, f_r)$ is the BASIS.

(2) $\Lambda^* := \{x \in V \mid \langle x, \lambda\rangle \in \mathbb{Z} \quad \forall\, \lambda \in \Lambda\}$ is the DUAL of Λ.

(3) Λ is INTEGRAL iff $\forall\, \lambda, \mu \in \Lambda : \langle \lambda, \mu\rangle \in \mathbb{Z}$.
It is EVEN iff $\forall\, \lambda \in \Lambda : \langle \lambda, \lambda\rangle \in 2\mathbb{Z}$.
It is SELFDUAL or UNIMODULAR iff $\Lambda^* = \Lambda$.

[7] In general, we call an extension of G by N central if N is in the center of $\tilde{G}$.

Exercise. Let $(V, \langle \cdot\,, \cdot \rangle)$, Λ, r as in Definition 1.11. Then show

- Λ is integral $\iff \Lambda \subset \Lambda^*$.
- Λ is even $\implies \Lambda$ is integral.
- If $r = n$, then Λ^* is a lattice.

Definition 1.12. Let $(V, \langle \cdot\,, \cdot \rangle)$ denote a real vectorspace of finite dimension n together with a non-degenerate symmetric bilinear form.
Two lattices Λ, Λ' are said to be isomorphic iff

$$\exists\, A \in \mathrm{O}(V, \langle \cdot\,, \cdot \rangle) : A\Lambda = \Lambda'$$

The automorphisms of a lattice are defined as

$$\mathrm{Aut}(\Lambda) := \{A \in \mathrm{O}(V, \langle \cdot\,, \cdot \rangle) \mid A\Lambda = \Lambda\}$$

Exercise. Let $(V, \langle \cdot\,, \cdot \rangle)$ as in Definition 1.11. Show:

(1) If $\langle \cdot\,, \cdot \rangle$ is positive definite and $\Lambda \subset V$ is a lattice of maximal rank, then $\mathrm{Aut}(\Lambda)$ is finite.

(2) For any lattice $\Lambda \subset V$, $\mathrm{Aut}(\Lambda)$ is simple iff $\mathrm{Aut}(\Lambda) \simeq \mathbb{Z}_2$. Hint: look for a subgroup of order 2.

Example 1.13. Let $V := \mathbb{R}^{2D}, D \in \mathbb{N} \setminus \{0\}$. Define a bilinear form

$$\langle x, y \rangle := \sum_{j=1}^{2D} (-1)^{j+1} x_j y_j \quad \forall\, x, y \in V$$

Apparently, the Gram-Matrix with respect to the standard basis $(e_1, \ldots, e_{2D})$ is given by $\mathrm{diag}(1, -1, 1, -1, \ldots)$. Let us consider another basis: $(f_1^+, f_1^-, f_2^+, f_2^-, \ldots, f_D^+, f_D^-)$, where

$$f_j^\pm := \frac{1}{\sqrt{2}}(e_{2j-1} \pm e_{2j}), \quad \forall\, j \in \{1, \ldots, D\}$$

We can build a lattice from this basis:

$$\mathrm{II}_{D,D} := \left\{ \sum_{j=1}^{D} \left(a_j^+ f_j^+ + a_j^- f_j^- \right) \,\middle|\, a_j^\mp \in \mathbb{Z} \right\}$$

Now, the Gram-Matrix of $\langle \cdot\,, \cdot \rangle$ with respect to $(f_1^+, f_1^-, f_2^+, f_2^-, \ldots, f_D^+, f_D^-)$ is apparently

$$\mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots \right)$$

Thus, $\mathrm{II}_{D,D} = U^D$, where $U = \mathrm{II}_{1,1}$ is the hyperbolic lattice. Two facts are worth noting:

- $\mathrm{II}_{D,D}$ is an even lattice, since the quadratic form induced by $\langle \cdot\,, \cdot \rangle$ is even on the basis and $\langle \cdot\,, \cdot \rangle$ is integral on $\mathrm{II}_{D,D}$.
- $\mathrm{II}_{D,D}^* \ni x = \sum_{j=1}^{D} \left(a_j^+ f_j^+ + a_j^- f_j^- \right) \iff \forall\, j : \mathbb{Z} \ni \left\langle x, f_j^\pm \right\rangle = a_j^\pm \iff x \in \mathrm{II}_{D,D}$
  In other words: $\mathrm{II}_{D,D} = \mathrm{II}_{D,D}^*$ is selfdual.

Exercise. Let $D \in \mathbb{N} \setminus \{0\}$ Show that $\mathrm{Aut}(\mathrm{II}_{D,D})$ is finite iff $D = 1$.

Theorem 1.14. (Milnor's Theorem) [Milnor 1958, Serre 1962]
Let $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$ denote a real vectorspace of finite dimension n together with a non-degenerate symmetric bilinear form of signature[8] (d_+, d_-).

(1) *If an even selfdual lattice $\Lambda \subset V$ of rank n exists, then*

$$d_+ \equiv d_- \mod 8$$

(2) *If $d_+ > 0$, $d_- > 0$ and $d_+ \equiv d_- \mod 8$, then there is a — up to isomorphism — unique self dual lattice $\Lambda \subset V$ of rank n.*

Theorem 1.15. [Niemeier '73]
There are precisely 24 isomorphism classes of even selfdual lattices with rank 24 in Euklidian $\mathbb{R}^{24}$.
If Λ is such a lattice, then $R_\Lambda := \{\lambda \in \Lambda \mid \langle\lambda, \lambda\rangle = 2\}$ obeys the following: Either $R_\Lambda = \varnothing$ or $\mathbb{R}_\Lambda$ contains a basis of $\mathbb{R}^{24}$.
Even selfdual lattices Λ, $\Lambda' \subset \mathbb{R}^{24}$ of rank 24 are isomorphic iff $\exists\, A \in \mathrm{O}(24) : R_{\Lambda'} = A\, R_\Lambda$.
There are such lattices with $R_\Lambda = \varnothing$.

Definition 1.16. [Leech 1965] The even selfdual lattices of rank 24 in Euklidian $\mathbb{R}^{24}$ are called Niemeier lattices.
For such a lattice Λ, R_Λ is called a root system and the elements $\alpha \in R_\Lambda$ are called roots.
If $R_\Lambda = \varnothing$, then Λ is called "Leech lattice" and is denoted as Λ_{Leech}.

Exercise. Consider a Niemeier lattice Λ and a root $\alpha \in R_\Lambda$. Then show:
The reflection S_α along $\alpha^\perp$ is an automorphism of Λ and $S_\alpha R_\Lambda = R_\Lambda$.

For a construction of the Leech lattice see [Conway/Sloane93].

Note. Since every lattice is of course an Abelian group $(\Lambda, +)$, we can apply the theory of extensions summarized above. In particular, we can consider central extensions of Λ by $\mathbb{Z}_2$:

$$1 \to \mathbb{Z}_2 \overset{\iota}{\hookrightarrow} \tilde{\Lambda} \overset{\pi}{\twoheadrightarrow} \Lambda \to 1$$

Exercise. If (α, ν) is a parameter system of such a central extension, then α is trivial:

$$\forall\, \lambda \in \Lambda : \alpha_\lambda = \mathrm{id}_{\mathbb{Z}_2}$$

Theorem 1.17. [Frenkel/Kac 1980]
Let $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$ be as in Definition 1.11 and Λ an integral lattice of rank $n = \dim_{\mathbb{R}} V$. Then there exists precisely one equivalence class of parameter systems of Λ in $\mathbb{Z}_2$ that contains a parameter system (α, ν) with $\alpha_\lambda = \mathrm{id}_{\mathbb{Z}_2}\ \forall\, \lambda \in \Lambda$ such that $\nu(0, \lambda) = \nu(\lambda, 0) = 1\ \forall\, \lambda \in \Lambda$ and

$$\nu(\lambda, \mu) = \nu(\mu, \lambda) \cdot (-1)^{\langle\lambda,\lambda\rangle\langle\mu,\mu\rangle + \langle\lambda,\mu\rangle} \quad \forall\, \lambda, \mu \in \Lambda \tag{$**$}$$

Remark. $(**)$ is natural in the context of VOAs.

Note. If Λ has rank 24, then $\Lambda/2\Lambda \simeq \mathbb{Z}_2^{24}$. The class of cocycles obtained in Theorem 1.17 for integral lattices Λ induces the choice of a cocycle ν that governs a central extension of $\mathbb{Z}_2^{24} = \Lambda/2\Lambda$ by $\mathbb{Z}_2$.
If Λ is even and selfdual, then this extension is nontrivial.

Exercise. Show, that in the setting of the note above, $\iota(\mathbb{Z}_2)$ *is* the center of the centrally extended group.

[8] For signatures we use the shorthand notation (d_+, d_-) instead of $(\underbrace{1, \ldots, 1}_{d_+\text{ times}}, \underbrace{-1, \ldots, -1}_{d_-\text{ times}})$.

Definition 1.18.

(1) The extra special group obtained by centrally extending

$$\mathbb{Z}_2^{24} \simeq {}^{\Lambda_{\text{Leech}}}\!\big/_{2\Lambda_{\text{Leech}}}$$

by the cocycle solving $(**)$ in Theorem 1.17 is called 2_+^{1+24}.
Cf. [Dixon/Ginsparg/Harvey88]

(2) $\mathrm{Co}_0 := \mathrm{Aut}(\Lambda_{\text{Leech}})$ is the 0^{th} Conway group.

(3) $\mathrm{Co}_1 := {}^{\mathrm{Co}_0}\!/_{\{\pm \mathrm{id}\}}$

Remark 1.19.

- Co_1 is a simple sporadic group. It has an extension by 2_+^{1+24}

$$C = 2_+^{1+24}.\,\mathrm{Co}_1$$

  which is a maximal subgroup of $\mathbb{M}$. (C is the centralizer of an element of order 2 in $\mathbb{M}$; $\mathbb{M}$ can be generated by C and another element of order 2 in $\mathbb{M}$.)

- If $2_+^{1+24} \hookrightarrow \mathrm{GL}_N(\mathbb{C})$ is a homomorphism with N minimal, then $N = 2^{12}$.

- The Leech lattice also speaks to the other side of Monstrous Moonshine:

$$\sum_{\lambda \in \Lambda_{\text{Leech}}} q^{\frac{\langle \lambda, \lambda \rangle}{2}} = (J(\tau) + 24)\, q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

06 Nov 2017

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2 Lie Algebras

2.1 Basics

Definition 2.1.1. Let k denote a field. A LIE ALGEBRA over k is a k-vectorspace $\mathfrak{g}$ with a LIE BRACKET

$$[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (x, y) \mapsto [x, y]$$

i.e. $[\,\cdot\,,\,\cdot\,]$ is bilinear, antisymmetric and satisfies the JACOBI IDENTITY

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \quad \forall\, x, y, z \in \mathfrak{g}$$

If $[x, y] = 0$ for all $x, y \in \mathfrak{g}$, then the Lie algebra is ABELIAN.
For two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ a HOMOMORPHISM is a k-linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$\phi([x, y]_1) = [\phi(x), \phi(y)]_2 \quad \forall\, x, y \in \mathfrak{g}_1$$

Example 2.1.2. For $N \in \mathbb{N} \setminus \{0\}$ let $\mathfrak{g} := \mathrm{Mat}_k(N \times N)$ and consider the commutator:

$$[X, Y] = XY - YX \quad \forall\, X, Y \in \mathfrak{g}$$

Then $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$ is a Lie algebra and so is any subspace of $\mathfrak{g}$ which is closed under $[\,\cdot\,,\,\cdot\,]$, e.g

$$\left\{A \in \mathfrak{g} \,\middle|\, A = -A^\top\right\}, \{A \in \mathfrak{g} \mid \operatorname{tr} A = 0\}$$

The verification is left as an exercise.
More abstractly, $\mathfrak{g} = \mathrm{End}_k(V)$ for any k-vectorspace V with $[X, Y] = X \circ Y - Y \circ X$ gives a Lie algebra.

Remark 2.1.3. If G is a Lie group, (i.e. a group with the smooth structure of a manifold such that for any $g \in G$ the map $G \to G : h \mapsto gh$ is smooth, as is $G \to G : h \mapsto h^{-1}$) then $\mathfrak{g} = T_{\mathrm{id}}G$ carries the structure of a Lie algebra.
For the definition of the Lie bracket use

$$G \times \mathfrak{g} \to TG : (g, X) \mapsto g.X := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (g\,\gamma(t))$$

where $\gamma : (-\varepsilon, \varepsilon) \to G$ is a smooth curve with $\gamma(0) = \mathrm{id}$ and $\dot\gamma(0) = X$. This defines a LEFTINVARIANT VECTORFIELD

$$\tilde X_g = g.X \quad \forall\, g \in G$$

Now use the Lie bracket on $\Gamma(TG)$

$$[\tilde X, \tilde Y] = \tilde X \circ \tilde Y - \tilde Y \circ \tilde X$$

where $\tilde X, \tilde Y$ are considered as linear maps $C^\infty(G) \to C^\infty(G)$, and set $[X, Y] := [\tilde X, \tilde Y]_{\mathrm{id}}$.
Now one needs to check that this does indeed define a Lie bracket on $\mathfrak{g}$.

Note. Let G denote a matrix Lie group, for instance $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}(n)$, i.e. a group $G \subset \mathrm{Mat}_{\mathbb{R}}(n \times n)$ with the smooth structure of a submanifold. Then the tangent space at a point $g \in G$ is

$$T_gG = \left\{\dot A(0) \,\middle|\, A : (-\varepsilon, \varepsilon) \to G \text{ smooth, } A(0) = g\right\} \subset \mathrm{Mat}_{\mathbb{R}}(n \times n)$$

and all the machinery in Remark 2.1.3 simplifies to the multiplication of matrices:

$$[X, Y] = X \cdot Y - Y \cdot X \quad \forall\, X, Y \in T_{\mathrm{id}}G$$

In practice, many of our examples will arise from $\mathfrak{g}_{\mathbb{R}} = T_{\mathrm{id}}G$ and $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$.

Example 2.1.4.

(1) Considering $G = \mathrm{SL}_2(\mathbb{C})$, we find that the associated Lie algebra takes the form

$$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \{X \in \mathrm{Mat}_{\mathbb{C}}(2\times 2) \mid \mathrm{tr}\, X = 0\}$$

To see this, consider any smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{SL}_2(\mathbb{C})$ with $\gamma(0) = \mathrm{id}$. Let $\lambda_{1,2}(t)$ denote the eigenvalues of $\gamma(t)$. Now, $\lambda_{1,2}$ are smooth in a neighborhood of 0 and $\lambda_1(t)\cdot\lambda_2(t) = 1$. Thus:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\lambda_1(t)\lambda_2(t)) \\ &= \dot\lambda_1(0)\lambda_2(0) + \lambda_1(0)\dot\lambda_2(0) \\ &= \dot\lambda_1(0) + \dot\lambda_2(0) = \mathrm{tr}\,\dot\gamma(0) \end{aligned}$$

To see that the right hand side is contained in $\mathfrak{sl}_2(\mathbb{C})$, one easily verifies that for all $X \in \mathrm{Mat}_{\mathbb{C}}(2\times 2)$ with $\mathrm{tr}\, X = 0$, the curve

$$\gamma(t) = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}$$

lies in $\mathrm{SL}_2(\mathbb{C})$ and satisfies $\gamma(0) = \mathrm{id}$, $\dot\gamma(0) = X$.

(2) More generally, similar results apply to $G = \mathrm{SL}_n(\mathbb{C})$:

$$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \{X \in \mathrm{Mat}_{\mathbb{C}}(n\times n) \mid \mathrm{tr}\, X = 0\}$$

The proof is left as an exercise. Hint: Use the following basis of $\mathfrak{g}$:

$$H_a \text{ for } a = 1, \dots, n-1,\ E_{kl}, E_{lk} \text{ for } 1 \le l < k \le n$$

$$\text{where}\quad H_a = \mathrm{diag}(0, \dots, 0, \underset{\substack{\uparrow\\ a}}{1}, \underset{\substack{\uparrow\\ a+1}}{-1}, 0, \dots, 0) \quad\text{and}\quad E_{kl} = e_k \otimes (e_l)^*$$

Example 2.1.5. Consider a $\mathbb{C}$-vectorspace $\mathfrak{g}$ with basis $(l_n)_{n\in\mathbb{Z}}$ and the bracket

$$\forall n, m \in \mathbb{Z} : [l_n, l_m] = (m-n)l_{m+n}$$

which is defined on all of $\mathfrak{g}$ by bilinear extension.
As an exercise, one can show that this yields an infinite dimensional Lie algebra. It is called the Witt algebra[1].

Definition 2.1.6. If $\mathfrak{g}$ is a Lie algebra and and $\mathfrak{h} \subset \mathfrak{g}$ is another Lie algebra such that $\mathfrak{h}$ inherits its vectorspace-structure and its Lie bracket from $\mathfrak{g}$, then $\mathfrak{h}$ is called a Lie subalgebra of $\mathfrak{g}$.

Theorem 2.1.7. (Ado's Theorem) [Ado 1935]
Every finite dimensional Lie algebra over a field of characteristic zero is (isomorphic to) a Lie subalgebra of some $\mathrm{Mat}_k(N\times N)$ *with the commutator as the Lie bracket.*

Theorem 2.1.8. (Lie's Third Theorem) [Lie 1888, 90, 93, Cartan 1930]
Every finite dimensional Lie algebra over $\mathbb{R}$ *is isomorphic to some* $T_{\mathrm{id}}G$ *where* G *is a connected Lie group. There is always a unique simply connected* G *with this property and there is alway one that can be given as a matrix group.*

[1] Ernst Witt, 1911 - 1991, doctoral student of Emmy Noether's

Definition 2.1.9. Let $\mathfrak{g}$ denote a Lie algebra over k with Lie bracket $[\cdot, \cdot]$.

(1) A vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is an IDEAL of $\mathfrak{g}$ iff

$$[\mathfrak{g}, \mathfrak{h}] := \operatorname{span}_k \{[X, H] \mid X \in \mathfrak{g}, H \in \mathfrak{h}\} \overset{!}{\subset} \mathfrak{h}$$

(2) $\mathfrak{g}$ is SIMPLE iff its only ideals are $\{0\}$ and $\mathfrak{g}$, and $\mathfrak{g}$ is not Abelian.

(3) $\mathfrak{g}$ is SEMISIMPLE[2] iff it is the direct sum[3] of simple Lie algebras $(\mathfrak{g}^{(a)})_{a\in I}$ over k.

$$\mathfrak{g} = \bigoplus_{a\in I} \mathfrak{g}^{(a)} \quad \text{where} \quad \left[\mathfrak{g}^{(a)}, \mathfrak{g}^{(a)}\right] = \{0\}$$

Exercise.

- Ideals are Lie subalgebras.
- If $\mathfrak{g}$ is a simple Lie algebra, then $\mathfrak{g}$ has trivial center

$$C(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \ \forall\, y \in \mathfrak{g}\}$$

  and its dimension is at least two.
- $\mathfrak{sl}_n(\mathbb{C})$ is a simple Lie algebra. (see Example 2.1.4)
- The Witt algebra (Example 2.1.5) is simple.

Remark 2.1.10.

(1) Let $\mathfrak{g} = T_{\mathrm{id}}G$ for a Lie group G, then $\mathfrak{h} \subset \mathfrak{g}$ is an ideal iff $\mathfrak{h} = T_{\mathrm{id}}N$ for some Lie subgroup N with $N \lhd G$.

(2) If $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ [4].

Definition 2.1.11. Consider a Lie algebra $\mathfrak{g}$ over k.

(1) A REPRESENTATION of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\varrho : \mathfrak{g} \to \operatorname{End}_k(V)$$

for some k-vectorspace V where $\operatorname{End}_k(V)$ is equipped with the commutator as a Lie bracket. We say $\mathfrak{g}$ ACTS on V; V is a "$\mathfrak{g}$-MODULE".

(2) For $V = \mathfrak{g}$, the ADJOINT REPRESENTATION is defined as the following map:

$$\mathrm{ad} : \mathfrak{g} \to \operatorname{End}_k(V) : X \mapsto \mathrm{ad}_X := [X, \cdot\,]$$

(3) If $\dim_k(\mathfrak{g}) < \infty$, then the map

$$\varkappa : \mathfrak{g} \times \mathfrak{g} \to k : (X, Y) \mapsto \operatorname{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$$

is called the KILLING FORM[5].

[2] In the literature, a different definition is frequently used, which for finite dimensional Lie algebras is equivalent and more general, otherwise.

[3] If we consider a direct sum of Lie algebras, we explicitly say so. Any other direct sums (even if the summands themselves are Lie algebras) are to be understood as direct sums in the sense of vectorspaces.

[4] The converse (though claimed in [Gannon98]) is not true.

[5] Wilhelm Killing (1847 - 1923) discovered the Killing form in the year of his death and discovered Lie algebras independently of Sophus Lie. His doctoral advisors were Karl Weierstraß and Ernst Eduard Kummer.

Remark 2.1.12.

(1) The adjoint representation is indeed a representation. (exercise!)

(2) If $\mathfrak{g}$ is finite dimensional, then

(a) The Killing form is bilinear, symmetric and '$\mathfrak{g}$-invariant', with the latter meaning that $\forall\, X, Y, Z \in \mathfrak{g}$:

$$\varkappa(X, [Y, Z]) = \varkappa([X, Y], Z) \iff \varkappa(\mathrm{ad}_Y(X), Z) = -\varkappa(X, \mathrm{ad}_Y(Z))$$

08 Nov 2017 (b) If $\mathrm{char}(k) = 0$, then Cartan's Criterion[6] holds: $\mathfrak{g}$ is semisimple iff $\varkappa$ is non-degenerate.

2.2 General structure theory for finite dimensional Lie algebras over $\mathbb{C}$

In this entire subsection, $\mathfrak{g}$ denotes a finite dimensional Lie algebra over $\mathbb{C}$.

Example 2.2.1. (for the notation, see Example 2.1.4)

(1) Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and consider the basis (h, e, f), where

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It easily checked, that $\mathrm{ad}_h(h) = [h, h] = 0$, $\mathrm{ad}_h(e) = [h, e] = 2e$ and $\mathrm{ad}_h(f) = [h, f] = -2f$. Thus, h, e and f are eigenvectors of ad_h with eigenvalues $0, 2$ and -2, respectively, and

$$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}h \oplus \mathbb{C}e \oplus \mathbb{C}f$$

is the eigenspace decomposition with respect to ad_h.

(2) As an exercise, consider $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and show: if

$$\mathfrak{h} := \mathrm{span}_{\mathbb{C}} \{H_a \mid a \in \{1, \dots, n-1\}\},$$
$$\mathfrak{g}^+ := \bigoplus_{k<l} \mathbb{C}E_{kl} \quad \text{and} \quad \mathfrak{g}^- := \bigoplus_{k>l} \mathbb{C}E_{kl},$$

then for any $H \in \mathfrak{h}$ the direct sum

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^+ \oplus \mathfrak{g}^-$$

is an eigenspace decomposition with regard to ad_H. We introduce the following notation for the eigenvalue of ad_H with eigenvector E_{kl} for some $k \neq l$:

$$\mathrm{ad}_H(E_{kl}) = \underbrace{\alpha_{kl}(H)}_{\in\mathbb{C}} E_{kl}$$

Then show that for all $a \in \{1, \dots, n-1\}$

$$\alpha_{kl}(H_a) = \begin{cases} +2, & \text{if } a = k = l-1 \\ -2, & a = l = k-1 \\ +1, & a = k \quad \text{or} \quad a = l-1, k \neq l-1 \\ -1, & a = l \quad \text{or} \quad a = k-1, l \neq k-1 \\ 0, & \text{otherwise} \end{cases} \qquad (*)$$

and $\alpha_{kl} : \mathfrak{h} \to \mathbb{C}$ is linear, so $\alpha_{kl} \in \mathfrak{h}^*$

[6] introduced in his dissertation (1894, with his doctoral advisors being Sophus Lie and Gaston Darboux) by Élie Cartan (1869 - 1951).

Definition 2.2.2. A Lie algebra $\mathfrak{h}$ is called NILPOTENT iff $\exists N \in \mathbb{N} : \mathfrak{h}^N = \{0\}$, where $\mathfrak{h}^0 := \mathfrak{h}$ and $\mathfrak{h}^{n+1} := [\mathfrak{h}^n, \mathfrak{h}]$ for any $n \in \mathbb{N}$.

Note. An alternative notation is $\mathfrak{h}^j = (\mathrm{ad}_\mathfrak{h})^j(\mathfrak{h})$. This notation emphasizes that any Lie algebra is nilpotent iff ad_X is nilpotent (on $\mathfrak{h}$) for all $X \in \mathfrak{h}$. It is then also apparent that $\mathfrak{h}^0 \supset \mathfrak{h}^1 \supset \mathfrak{h}^2 \supset \cdots$. The Lie subalgebra $\mathfrak{h}^j \subset \mathfrak{h}$ is an ideal of $\mathfrak{h}$ for any $j \in \mathbb{N}$.

Example. For an arbitrary Lie algebra $\mathfrak{g}$, any Abelian Lie subalgebra is nilpotent, so in particular, with notations as in Example 2.2.1, $\mathfrak{h} \subset \mathfrak{sl}_n(\mathbb{C})$ is a nilpotent Lie subalgebra.

Proposition 2.2.3. *Consider a Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ a nilpotent Lie subalgebra. Then*

$$\mathfrak{h} \subset \left\{x \in \mathfrak{g} \,\middle|\, \forall H \in \mathfrak{h} : \exists N \in \mathbb{N} : (\mathrm{ad}_H)^N(x) = 0\right\} =: \mathrm{Hau}_0(\mathrm{ad}_\mathfrak{h})$$

Proof. For any $X \in \mathfrak{h}, H \in \mathfrak{h}$

$$(\mathrm{ad}_H)^j(X) \in \mathfrak{h}^j$$

holds for arbitrary positive integers j. Since $\mathfrak{h}$ is nilpotent, $\mathfrak{h}^N = \{0\}$ for sufficiently large N. Thus, we get $X \in \mathrm{Hau}_0(\mathrm{ad}_\mathfrak{h})$. □

Note. As an exercise, it may be shown that $\mathrm{Hau}_0(\mathrm{ad}_\mathfrak{h})$ as in Proposition 2.2.3 is a Lie subalgebra of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, Theorem 2.2.6 will show that

$$\mathfrak{h} = \mathrm{Hau}_0(\mathrm{ad}_\mathfrak{h})$$

for $\mathfrak{h}$ as in Example 2.2.1.

Theorem 2.2.4. (Structure Theorem A)
Let $\mathfrak{g}$ denote a finite dimensional Lie algebra over $\mathbb{C}$ and $\mathfrak{h} \subset \mathfrak{g}$ a nilpotent Lie subalgebra. Define[7] *for any $\alpha \in \mathfrak{h}^*$*

$$\begin{aligned}\mathfrak{g}_\alpha &:= \left\{X \in \mathfrak{g} \,\middle|\, \exists N \in \mathbb{N} : \forall H \in \mathfrak{h} : (\mathrm{ad}_H - \alpha(H)\,\mathrm{id})^N(X) = 0\right\}\\ &\quad (=: \mathrm{Hau}_\alpha(\mathrm{ad}_\mathfrak{h}))\end{aligned}$$

Then the following holds:

(1) *There is a finite set $\tilde{\Delta} \subset \mathfrak{h}^*$ such that*

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha = \bigoplus_{\alpha \in \tilde{\Delta}} \mathfrak{g}_\alpha$$

(2) *If $\alpha, \beta \in \mathfrak{h}^*$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ holds true*[8].

Proof. (Sketch)

(1) The first equality is actually a generalization of Jordan's decomposition theorem. The finiteness of $\tilde{\Delta}$ then follows since $\dim \mathfrak{g} < \infty$.

(2) Given $\alpha, \beta \in \mathfrak{h}^*$, consider $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$. Then show by induction that for any $H \in \mathfrak{h}$ and $N \in \mathbb{N}$:

$$\begin{aligned}&(\mathrm{ad}_H - (\alpha(H) + \beta(H))\,\mathrm{id})^N\,[X, Y] =\\ &\qquad \sum_{j=0}^{N} \binom{N}{j} \left[(\mathrm{ad}_H - \alpha(H)\mathrm{id})^j(X), (\mathrm{ad}_H - \beta(H)\mathrm{id})^{N-j}(Y)\right]\end{aligned}$$

Note that if $N \geq 2\dim\mathfrak{g}$, for any index j in the sum above, at least one of the two operands of the Lie bracket vanishes in every summand.

[7] $\mathrm{Hau}_\alpha(\mathrm{ad}_\mathfrak{h}) = \bigcap_{H \in \mathfrak{h}} \mathrm{Hau}_{\alpha(H)}(\mathrm{ad}_H)$

[8] In other words: $\mathfrak{g}$ has a $\mathfrak{h}^*$-grading.

□

Definition 2.2.5. If $\mathfrak{g}$ is a Lie algebra, then a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called CARTAN SUBALGEBRA iff $\mathfrak{h}$ is nilpotent and it agrees with its NORMALIZER[9]:

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid [X, H] \in \mathfrak{h} \;\forall H \in \mathfrak{h}\}$$

Note. For $\mathfrak{sl}_n(\mathbb{C})$, the subalgebra $\mathfrak{h}$ from Example 2.2.1 is indeed a Cartan subalgebra:

- It is Abelian and therefore nilpotent.
- It agrees with its normalizer $\{X \in \mathfrak{g} \mid [X, H] \in \mathfrak{h} \;\forall H \in \mathfrak{h}\}$, as $(*)$ from Example 2.2.1 implies.

In fact, $\mathfrak{h}$ is a MAXIMAL ABELIAN LIE SUBALGEBRA (as is any Cartan subalgebra of a semisimple Lie algebra).
Here $\mathfrak{h} = \mathrm{Hau}_0(\mathrm{ad}_X)$ for any diagonal $X \in \mathfrak{sl}_n(\mathbb{C})$ whose eigenvalues are pairwise distinct. Even more, for any $Y \in \mathfrak{sl}_n(\mathbb{C})$ with pairwise distinct eigenvalues $\mathrm{Hau}_0(\mathrm{ad}_Y)$ is a Cartan subalgebra and[10]

$$\mathrm{Hau}_0(\mathrm{ad}_Y) = S\,\mathfrak{h}\,S^{-1} \quad \text{for some } S \in \mathrm{GL}_n(\mathbb{C}).$$

Such Y are called REGULAR. More generally, for any finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, $X \in \mathfrak{g}$ is called regular iff $\mathrm{Hau}_0(\mathrm{ad}_X)$ has the minimal possible dimension. One then shows, that if in addition ad_X is diagonalizable, then $\mathrm{Hau}_0(\mathrm{ad}_X)$ is a Cartan subalgebra.

Theorem 2.2.6. (Structure Theorem B)
For any finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, the following holds:

(1) *There exists a Cartan subalgebra.*

(2) *If $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, then*

 (a) $\mathfrak{h} = \mathrm{Hau}_0(\mathrm{ad}_{\mathfrak{h}})$

 (b) *There is a finite set $\tilde{\Delta} \in \mathfrak{h}^*$ such that*

$$\mathfrak{g} = \bigoplus_{\alpha \in \tilde{\Delta}} \mathfrak{g}_\alpha \quad \textit{where} \quad \mathfrak{g}_\alpha = \mathrm{Hau}_\alpha(\mathrm{ad}_{\mathfrak{h}})$$

 For any $\alpha, \beta \in \mathfrak{h}^*$, the inclusion $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ holds.

Proof. (Sketch)

(1) is contained in the note above, since there always exists a regular $X \in \mathfrak{g}$ with diagonalizable ad_X.

(2) (a) The inclusion "$\subset$" is the subject of Proposition 2.2.3. We show "$\supset$" by contradiction: Assume there is some $Y \in \mathrm{Hau}_0(\mathrm{ad}_{\mathfrak{h}})$ such that $Y \notin \mathfrak{h}$. Because $\mathfrak{h}$ is a Cartan subalgebra, it agrees with its normalizer and hence contains some H such that $[H, Y] = \mathrm{ad}_H(Y) \notin \mathfrak{h}$.
On the other hand, $Y \in \mathrm{Hau}_0(\mathrm{ad}_{\mathfrak{h}})$ holds. Thus, $(\mathrm{ad}_H)^N(Y) = 0 \in \mathfrak{h}$ holds for sufficiently large $N \in \mathbb{N}$. In particular, there is some $n \in \mathbb{N}$ such that $X := (\mathrm{ad}_H)^n(Y) \notin \mathfrak{h}$, but $\mathrm{ad}_H(X) = (\mathrm{ad}_H)^{n+1}(Y) \in \mathfrak{h}$. In other words: X is in the normalizer of $\mathfrak{h}$ but not in $\mathfrak{h}$, violating the premiss that $\mathfrak{h}$ is a Cartan subalgebra.

 (b) we just copied from Theorem 2.2.4 to have it all in one place.

□

[9] In [Carter/Segal/McDonald95], this is called the IDEALIZER, because it is the largest Lie subalgebra of $\mathfrak{g}$ of which $\mathfrak{h}$ is an ideal.
[10] That is, $\mathfrak{h}$ and $\mathrm{Hau}_0(\mathrm{ad}_Y)$ agree up to INNER AUTOMORPHISM.

Definition 2.2.7. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$, then its RANK is

$$\operatorname{rk}(\mathfrak{g}) = \min \{\dim(\operatorname{Hau}_0(\operatorname{ad}_X)) \mid X \in \mathfrak{g}\}.$$

In particular $\operatorname{rk}(\mathfrak{g}) = \dim(\mathfrak{h})$ for any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, as was shown in the discussion preceding Theorem 2.2.6.

Theorem 2.2.8. *If $\mathfrak{g}$ is a finite dimensional complex Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ is as Cartan subalgebra, consider the generalized eigenspace decomposition from Theorem 2.2.4*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\Delta \subset \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq \{0\}$ for any $\alpha \in \Delta$. Keeping in mind that $\mathfrak{g}_0 = \mathfrak{h}$, we claim: if $\alpha, \beta \in \Delta \cup \{0\}$ with $\alpha + \beta \neq 0$, then $\operatorname{ad}_X \circ \operatorname{ad}_Y$ is nilpotent for all $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$. In particular, $\varkappa(X, Y) = 0$ holds.

Proof. Given X, Y as in the theorem, set $A := \operatorname{ad}_X \circ \operatorname{ad}_Y$. Theorem 2.2.4 yields

$$A^N(\mathfrak{g}_\gamma) \subset \mathfrak{g}_{N(\alpha+\beta)+\gamma} \quad \forall \gamma \in \Delta, N \in \mathbb{N}$$

Since $\mathfrak{g}$ is finite dimensional, we can choose $N \gg 0$ such that $\mathfrak{g}_{N(\alpha+\beta)+\gamma} = \{0\}$ for any $\gamma \in \Delta$. Thus, A is nilpotent and apparently $0 = \operatorname{tr}(A) = \varkappa(X, Y)$. □

Proposition 2.2.9. (Lie's Theorem) *If $\mathfrak{g}$ is a finite dimensional Lie algebra over $\mathbb{C}$ and $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, then (using the notations from Theorem 2.2.8) for any $\alpha \in \Delta \cup \{0\}$, the Lie subalgebra $\mathfrak{g}_\alpha$ has a basis with respect to which all ad_X (for $X \in \mathfrak{h}$) are represented by upper triangular matrices.*

The proof is a generalization of the analogous result in linear algebra.

Proposition 2.2.10. *With notations as in Theorem 2.2.8,*

$$\forall X, Y \in \mathfrak{h} : \varkappa(X, Y) = \sum_{\alpha \in \Delta} \alpha(X) \cdot \alpha(Y) \cdot \dim(\mathfrak{g}_\alpha)$$

Proof. Exercise. □

2.3 The semisimple case

Remark 2.3.1. For a finite dimensional complex Lie algebra, the following statements are equivalent[11]:

(1) $\mathfrak{g}$ is semisimple: it is a direct sum $\mathfrak{g} = \bigoplus_a \mathfrak{g}^{(a)}$ of Lie algebras with all $\mathfrak{g}^{(a)}$ being simple.

(2) $\mathfrak{g}$ has no nontrivial Abelian ideals.

(3) The Killing form $\varkappa$ is non-degenerate. (Cartan's criterion)

Remark. We provide some fragments of the proof of Remark 2.3.1:

"(1) $\implies$ (3)" follows since $\ker(\varkappa) = \{X \in \mathfrak{g} \mid \varkappa(X,Y) = 0 \;\forall Y \in \mathfrak{g}\}$ is an ideal in $\mathfrak{g}$.

"(3) $\implies$ (2)" follows since for any Abelian ideal $\mathfrak{a} \subset \mathfrak{g}$ the following holds: Considering any $A \in \mathfrak{a}, X \in \mathfrak{g}$,

$$\mathfrak{g} \xrightarrow{\mathrm{ad}_A} \mathfrak{a} \xrightarrow{\mathrm{ad}_X} \mathfrak{a} \xrightarrow{\mathrm{ad}_A} \{0\}\,.$$

Thus, $(\mathrm{ad}_A \circ \mathrm{ad}_X)^2 = 0$ follows, implying $\varkappa(A,X) = 0$ and (by (3)) $A = 0$.

Theorem 2.3.2. *Let $\mathfrak{g}$ denote a finite dimensional semisimple Lie algebra over $\mathbb{C}$ and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Consider the decomposition from Theorem 2.2.8: $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$. Then the following statements hold:*

(1) $\varkappa|_{\mathfrak{h}\times\mathfrak{h}}$ *is non-degenerate and* $\Delta = -\Delta$.

(2) $\mathfrak{h}$ *is Abelian.*

(3) Δ *generates* $\mathfrak{h}^*$ *as a vectorspace over* $\mathbb{C}$.

Proof.

(1) $\varkappa$ is non-degenerate on $\mathfrak{g}$ (2.3.1 (3), Cartan's criterion) and by Theorem 2.2.8, $\mathfrak{h} \perp_\varkappa \mathfrak{g}_\alpha$ holds for any $\alpha \in \Delta$. Thus $\varkappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate.
If $\alpha \in \Delta$, then by Theorem 2.2.8, $\mathfrak{g}_\beta \perp_\varkappa \mathfrak{g}_\alpha$ for any $\beta \neq -\alpha$. Since $\varkappa$ is non-degenerate, $\mathfrak{g}_{-\alpha} \neq \{0\}$ holds, implying $-\alpha \in \Delta$.

(2) Let $H_1, H_2 \in \mathfrak{h}$ and set $X := [H_1, H_2] \in \mathfrak{h} = \mathfrak{g}_0$. Then we notice two things:

- Given any $Y \in \mathfrak{g}_\alpha$ for some $\alpha \in \Delta$, Theorem 2.2.8 implies that $\varkappa(X,Y) = 0$.
- For any $H \in \mathfrak{h}$, $\varkappa(X,H) = 0$ holds, since $\mathrm{ad}_X = [\mathrm{ad}_{H_1}, \mathrm{ad}_{H_2}]$ and hence

$$\varkappa(X,H) = \mathrm{tr}([\mathrm{ad}_{H_1}, \mathrm{ad}_{H_2}] \circ \mathrm{ad}_H) \overset{!}{=} 0$$

where the last equality follows from Lie's Theorem 2.2.9 since $\mathrm{tr}(A_1A_2A_3) = \mathrm{tr}(A_2A_1A_3)$ for upper triangular matrices A_1, A_2, A_3.

Together with (1), this yields $X = 0$.

(3) We prove the last claim by contradiction:
Assuming the existence of $\beta \in \mathfrak{h}^* \setminus \mathrm{span}_{\mathbb{C}}(\Delta)$, we find some $H \in \mathfrak{h}$ such that $\beta(H) \neq 0$, but $\alpha(H) = 0$ for any $\alpha \in \Delta$. Thus $H \neq 0$ and $\mathrm{ad}_H = 0$. The latter equality implies $\varkappa(H,X) = 0 \;\forall X \in \mathfrak{g}$, violating the non-degeneracy of $\varkappa$.

□

[11] (1), (2) are both used as definitions.

Theorem 2.3.3. *With assumptions and notations as in Theorem 2.3.2, the following holds: For all $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ is one-dimensional and if $n\alpha \in \Delta$ for some $n \in \mathbb{Z}$, then $n \in \{\pm 1\}$.*

Proof. Exercise.
Hint: First show, that there are $X \in \mathfrak{g}_{-\alpha}$, $Y \in \mathfrak{g}_\alpha$ such that $H := [X, Y] \neq 0$, and that thus $\alpha(H) \neq 0$. Then consider $V := \operatorname{span}_{\mathbb{C}} \{X\} \oplus \mathfrak{h} \oplus \bigoplus_{n=1}^{\infty} \mathfrak{g}_{n\alpha}$ and study $\operatorname{tr}_V(\operatorname{ad}_H)$. □

In particular, by Theorem 2.3.3, the generalized eigenspaces $\mathfrak{g}_\alpha$ all are eigenspaces simultaneously for all $\operatorname{ad}_H, H \in \mathfrak{h}$.

Remark 2.3.4. Using the notations of Theorem 2.3.3, let $X \in \mathfrak{g}_{-\alpha}$, $Y \in \mathfrak{g}_\alpha$, $H = [X, Y]$ with $H \neq 0$. Then $\alpha(H) \neq 0$, as can be shown in the following fashion:
Given any $\beta \in \Delta$, consider the "α-string through β":

$$V_\beta := \bigoplus_{n=-\infty}^{\infty} \mathfrak{g}_{\beta+n\alpha}$$

Take $\operatorname{tr}_{V_\beta}(\operatorname{ad}_H) = \operatorname{tr}_{V_\beta}([\operatorname{ad}_X, \operatorname{ad}_Y]) = 0$ to deduce

$$n_\beta \beta(H) + m_\beta \alpha(H) = 0 \quad \text{for some } n_\beta, m_\beta \in \mathbb{Z}, n_\beta \neq 0 \tag{$*$}$$

Thus $\alpha(H) = 0$ leads to an immediate contradiction, as it implies that $\beta(H) = 0$ for any $\beta \in \Delta$, thereby infringing the non-degeneracy of $\varkappa$.
From $(*)$, we also easily obtain $\beta(H) \in \alpha(H) \cdot \mathbb{Q}$.
We will now use the above to construct a distinguished basis of $\mathfrak{h}$: for any pair $\pm\alpha \in \Delta$, we choose $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\varkappa(e_\alpha, f_\alpha) = 1$ and $h_\alpha := [e_\alpha, f_\alpha] \neq 0$. Then for any $H \in \mathfrak{h}$, we calculate

$$\begin{aligned} \varkappa(h_\alpha, H) &\overset{(2.1.12)}{=} -\varkappa(f_\alpha, [e_\alpha, H]) \\ &= \varkappa(f_\alpha, \alpha(H) e_\alpha) \\ &= \alpha(H) \end{aligned}$$

On the other hand, using the sum expansion from 2.2.10 yields

$$\alpha(h_\alpha) = \varkappa(h_\alpha, h_\alpha) = \sum_{\beta \in \Delta} (\beta(h_\alpha))^2 \overset{(*)}{=} (\alpha(h_\alpha))^2 q$$

for some $q \in \mathbb{Q}$. Thus, $\alpha(h_\alpha) \in \mathbb{Q}$ and by $(*)$ $\beta(h_\alpha) \in \mathbb{Q}$, so the sum in the equation above is non-negative, yielding $\varkappa(h_\alpha, h_\alpha) = \alpha(h_\alpha) > 0$, since $\alpha(h_\alpha) \neq 0$.
So if $(\alpha_1, \ldots, \alpha_r)$ with $\alpha_i \in \Delta \; \forall i$ forms a basis of $\mathfrak{h}^*$ (by Theorem 2.3.2), let

$$\mathfrak{h}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}} \{h_{\alpha_1}, \ldots, h_{\alpha_r}\}$$

Then $\mathfrak{h}_{\mathbb{R}}^* := \operatorname{Hom}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$ has a basis $(w_1, \ldots, w_r)$ with $w_j(h_{\alpha_i}) = \delta_{ij} \; \forall i, j \in \{1, \ldots, r\}$ and by the above $\Delta \subset \mathfrak{h}_{\mathbb{R}}^*$, so

$$\begin{aligned} \mathfrak{h}_{\mathbb{R}}^* &= \operatorname{span}_{\mathbb{R}}(\Delta) \\ \mathfrak{h}_{\mathbb{R}} &= \operatorname{span}_{\mathbb{R}} \{h_\alpha \mid \alpha \in \Delta\} \end{aligned}$$

Note.

- $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus \imath\, \mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}^* = \mathfrak{h}_{\mathbb{R}}^* \oplus \imath\, \mathfrak{h}_{\mathbb{R}}^*$ and by the above $\varkappa|_{\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}}$ is a Euklidean scalar product.

- $\mathfrak{sl}_\alpha := \mathrm{span}_\mathbb{C}\{h_\alpha, e_\alpha, f_\alpha\} \subset \mathfrak{g}$ defines a Lie subalgebra[12] with $\mathfrak{sl}_\alpha \simeq \mathfrak{sl}_2(\mathbb{C})$ (cf. Example 2.1.4).

Definition 2.3.5. For any complex finite dimensional semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, using the notations from Theorem 2.3.2

(1) $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ is called ROOT SPACE DECOMPOSITION, any $\alpha \in \Delta$ is called a ROOT and Δ is a ROOT SYSTEM.

(2) Let $(h_\alpha)_{\alpha\in\mathfrak{h}^*} \in \mathfrak{h}^{(\mathfrak{h}^*)}$ such that for all $\alpha \in \Delta$, $H \in \mathfrak{h}$, the relation $\varkappa(h_\alpha, H) = \alpha(H)$ holds. Then define

$$\mathfrak{h}^*_\mathbb{R} := \mathrm{span}_\mathbb{R}(\Delta)$$
$$\mathfrak{h}_\mathbb{R} := \mathrm{span}_\mathbb{R}\{h_\alpha \mid \alpha \in \Delta\}$$

the STANDARD REAL FORMS of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively.
For any $\alpha, \beta \in \mathfrak{h}^*_\mathbb{R}$, we define

$$\langle \alpha, \beta \rangle := \varkappa(h_\alpha, h_\beta)$$

Note. By Remark 2.3.4

- $\mathfrak{h}_\mathbb{R}, \mathfrak{g}_\mathbb{R}$ are Lie subalgebras[13] of $\mathfrak{g}$ viewed as a Lie algebra over $\mathbb{R}$.
- $\langle \cdot, \cdot \rangle$ defined in Definition 2.3.5 is positive definite and thus $(\mathfrak{h}^*_\mathbb{R}, \langle \cdot, \cdot \rangle)$ is a Euklidean vectorspace.

15 Nov 2017 **Theorem 2.3.6.** *Consider a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$ of rank r with Cartan subalgebra $\mathfrak{h}$. With notations as in Definition 2.3.5, the following holds:*

(1) $(\mathfrak{h}^*_\mathbb{R}, \langle \cdot, \cdot \rangle)$ *is a Euklidean vectorspace.*

(2) $\forall\, \alpha, \beta \in \Delta \cup \{0\}$: *if* $\alpha + \beta \neq 0$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$

(3) *There exists a decomposition* $\Delta = \Delta^+ \cup \Delta^-$ *such that* $\Delta^- = -\Delta^+$. *One can choose* $\alpha_1, \dots, \alpha_r \in \Delta^+$ *such that* $(\alpha_1, \dots, \alpha_r)$ *is a basis of* $\mathfrak{h}^*_\mathbb{R}$ *and for all* $\alpha \in \Delta^+$, *there exists* $(k_i) \in \mathbb{N}^r$ *so that* $\alpha = \sum_{i=1}^r k_i\alpha_i$.

(4) *For all* $\alpha \in \Delta$, *there exists a reflection along* $\alpha^\perp \subset \mathfrak{h}^*_\mathbb{R}$, *i.e.*

$$S_\alpha(x) = x - \frac{\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha \quad \forall\, x \in \mathfrak{h}^*_\mathbb{R}$$

yields $S_\alpha \in \mathrm{O}(\mathfrak{h}^*_\mathbb{R}, \langle \cdot, \cdot \rangle)$. *Moreover, the subgroup*

$$W := \langle S_\alpha \mid \alpha \in \Delta \rangle \subset \mathrm{O}(\mathfrak{h}^*_\mathbb{R}, \langle \cdot, \cdot \rangle)$$

is a finite group acting by permutation on Δ *and*

$$W\,\{\alpha_1, \dots, \alpha_r\} = \Delta.$$

For all $j \in \{1, \dots, r\}$, *the set* $\Delta \setminus \{\alpha_j\}$ *is invariant under* S_{α_j}.

(5) *For all* $i, j \in \{1, \dots, r\}$, *define*

$$a_{ij} := 2\frac{\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_i, \alpha_i\rangle}$$

Then the following holds:

[12] Thus, a root always gives rise to an 'embedding' $\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$.
[13] $\mathfrak{g}_\mathbb{R}$ is the real Lie algebra generated by the $e_\alpha, f_\alpha, h_\alpha$

(C1) $a_{jj} = 2 \quad \forall j$

(C2) $a_{ij} \in \mathbb{Z}\ \forall i,j$ *and* $a_{ij} \le 0$ *if* $i \ne j$

(C3)[14] $\forall i,j : a_{ij} = 0 \iff a_{ji} = 0$

(C4) $A = (a_{ij}) \in \mathrm{Mat}_{\mathbb{Z}}(r \times r)$ *is* SYMMETRIZABLE, *i.e. there is a matrix* $D = \mathrm{diag}(d_1, \dots, d_r)$ *such that* $(DA)^\top = DA$ *and* $d_1, \dots, d_r \ge 0$ *and* (DA) *is positive definite.*

Furthermore, given $e_j \in \mathfrak{g}_{\alpha_j}, f_j \in \mathfrak{g}_{-\alpha_j}$ *such that* $h_j := [e_j, f_j]$ *for* $j \in \{1, \dots, r\}$ *yields a basis* $(h_1, \dots, h_r)$ *of* $\mathfrak{h}_{\mathbb{R}}$*, we have*

(I)[15] $[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_j, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j$
for all $i, j \in \{1, \dots, r\}$ *(so* $a_{ij} = \alpha_j(h_i)$*)*

(II) $(\mathrm{ad}_{e_i})^{1-a_{ij}}(e_j) = (\mathrm{ad}_{f_i})^{1-a_{ij}}(f_j) = 0$ *for all* $i, j \in \{1, \dots, r\}$ *with* $i \ne j$

(6) *A is independent of all choices we made.*

Remark. (concerning the proof of Theorem 2.3.6)

(1) was proved in Remark 2.3.4.

(2) follows from a refinement of the discussion of $(*)$ in Remark 2.3.4:
Let $\alpha, \beta \in \Delta \cup \{0\}$ and consider the α-string through β,

$$V_\beta = \bigoplus_{n=-\infty}^{\infty} \mathfrak{g}_{\beta+n\alpha} = \bigoplus_{n=-N}^{M} \underbrace{\mathfrak{g}_{\beta+n\alpha}}_{\ne 0}$$

One checks that $M - N = 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ and also $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$ iff $\mathfrak{g}_{\alpha+\beta} = \{0\}$ from which $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ follows if $\alpha + \beta \ne 0$.

(3),(4) follow from a detailed study of the above structure.

(5) Just insert the observations already made into

$$e_j = 2\frac{e_{\alpha_j}}{\alpha_j(h_{\alpha_j})}, \quad f_j = f_{\alpha_j} \quad \text{and} \quad h_j = 2\frac{h_{\alpha_j}}{\alpha_j(h_{\alpha_j})}$$

(6) Follows from the uniqueness of $\mathfrak{h}$ up to inner automorphism.

Remark 2.3.7.

(1) $\dim \mathfrak{g} = \underbrace{\dim \mathfrak{h}}_{=\mathrm{rk}(\mathfrak{g})} + \underbrace{\#\Delta}_{\ge 2\,\mathrm{rk}(\mathfrak{g})} \ge 3\,\mathrm{rk}(\mathfrak{g})$

(2) A encodes $\mathfrak{g}$ entirely:

$$a_{ij} = 2\frac{\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_i, \alpha_i\rangle}$$

Thus $(\alpha_1, \dots, \alpha_r)$ can be obtained up to global scaling. Then $\Delta = W\{\alpha_1, \dots, \alpha_r\}$ and (I) & (II) yield $\mathfrak{g}$ as an abstract Lie algebra.

[14] Property (C3) follows from property (C4) but is listed nevertheless, since in many references, property (C4) is dropped until it is needed.

[15] $[h_i, h_j] = 0$ follows from the other relations but is listed nevertheless, by tradition.

(3) Using (2), we find a real from $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$ generated as a Lie algebra over $\mathbb{R}$ by

$$(e_1, \dots, e_r, f_1, \dots, f_r, h_1, \dots, h_r)$$

On $\mathfrak{g}_\mathbb{R}$, we define a linear map

$$\vartheta : \mathfrak{g}_\mathbb{R} \to \mathfrak{g}_\mathbb{R} : e_j \mapsto -f_j, f_j \mapsto -e_j, h_j \mapsto -h_j \quad \text{(+ linear extension)}$$

and notice that ϑ is indeed a Lie algebra automorphism on $\mathfrak{g}_\mathbb{R}$ and an involution[16]. One checks:

- $\vartheta([X,Y]) = [\vartheta X, \vartheta Y]$ implies $\varkappa(\vartheta X, \vartheta Y) = \varkappa(X,Y)$, since $\vartheta \circ \mathrm{ad}_X = \mathrm{ad}_{\vartheta X} \circ \vartheta$.
- anti-$\mathbb{C}$-linear extension of ϑ to $\mathfrak{g}$ yields a positive definite Hermitian sesquilinear form on $\mathfrak{g}$:

$$(X,Y) := -\varkappa(X, \vartheta Y)$$

  For example, for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \subset \mathrm{Mat}_\mathbb{C}(n \times n)$, we have $\vartheta(X) = \overline{X}^\top$.
  Such an involution (inducing a Hermitian product as above) is called a CARTAN INVOLUTION.

Remark. If ϑ is a Cartan involution on the real form $\mathfrak{g}_0$ of a finite dimensional semisimple complex Lie algebra,

$$\mathfrak{g}_0 = \mathrm{Eig}_{+1}(\vartheta) \oplus \mathrm{Eig}_{-1}(\vartheta)$$

is called a CARTAN DECOMPOSITION.

Definition 2.3.8. With notations as in Theorem 2.3.6, for any semisimple finite dimensional complex Lie algebra

(1) $\alpha \in \Delta^+$ are the POSITIVE ROOTS, $\alpha_1, \dots, \alpha_r \in \Delta^+$ are the SIMPLE or FUNDAMENTAL ROOTS. S_α is called WEYL REFLECTION and W is the WEYL GROUP.

(2) Any matrix $A = (a_{ij}) \in \mathrm{Mat}_\mathbb{Z}(r \times r)$ that obeys (C1) - (C4) from Theorem 2.3.6 is called CARTAN MATRIX.
The Lie algebra $\mathfrak{g}(A)$ generated by $e_1, \dots, e_r, f_1, \dots, f_r, h_1, \dots, h_r$ via the relations (I) and (II) from Theorem 2.3.6 is the ASSOCIATED LIE ALGEBRA.

(3) A Cartan matrix is called REDUCIBLE if, up to permutation of $\{1, \dots, r\}$, A is in block diagonal form. If A is not reducible, then we call it IRREDUCIBLE.

(4) If $A \in \mathrm{Mat}_\mathbb{Z}(r \times r)$ is a Cartan matrix, then the associated COXETER-DYNKIN DIAGRAM is the graph with vertices $V = \{1, \dots, r\}$ and $a_{ij}a_{ji}$ edges between the vertices i and j if $i \neq j$. The diagram may be refined to a mixed graph as follows: If $a_{ij} < a_{ji}$, the edges connecting i and j are directed from i to j.

Example. (cf. Example 2.2.1)
Consider, once again, $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ with Cartan subalgebra $\mathfrak{h} = \mathrm{span}_\mathbb{C}\{H_1, \dots, H_{n-1}\}$. From Example 2.2.1 it follows that $\mathrm{rk}(\mathfrak{g}) = r = n-1$ and that the roots are α_{kl} for $k, l \in \{1, \dots, n\}, k \neq l$. Furthermore, $\alpha_{kl} = -\alpha_{lk}$ and $\mathfrak{g}_{\alpha_{kl}} = \mathbb{C}E_{kl}$. As an exercise, show the following claims:

- One can choose $\Delta^+ := \{\alpha_{kl} \mid k < l\}$.
- The simple roots can be chosen as $\alpha_{j,j+1}$ for all $j \in \{1, \dots, n-1\}$. Then any positive root can be represented as

$$\alpha_{kl} = \sum_{j=k}^{l-1} \alpha_{j,j+1}$$

[16] i.e. $\vartheta \circ \vartheta = \mathrm{id}$

- The generators are then given as[17]

$$e_j = E_{j,j+1}, \quad f_j = E_{j+1,j} = e_j^\top$$

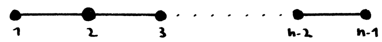
  and satisfy the conditions (I) and (II) from Theorem 2.3.6.

- If, for $i, j \in \{1, \ldots, n-1\}, i \neq j$ we set

$$a_{ii} = 2 \quad \text{and} \quad a_{ij} = \alpha_j(h_i) = \begin{cases} -1, & \text{if } |i-j| = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{i.e. } A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

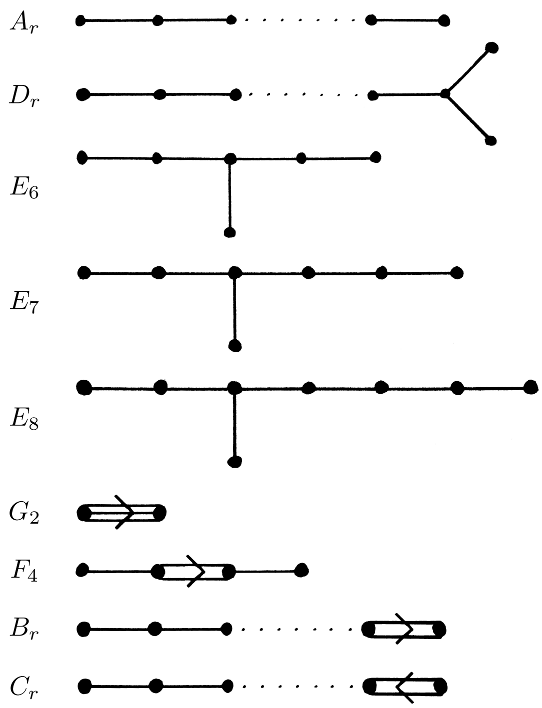
  then (C1) - (C4) from Theorem 2.3.6 hold and the Coxeter-Dynkin diagram is of "type A_r":



Theorem 2.3.9. (Classification of simple Lie algebras) [Serre 1966] *20 Nov 2017*
Let $A \in \mathrm{Mat}_{\mathbb{Z}}(r \times r)$ denote an irreducible Cartan matrix and $\mathfrak{g}(A)$ the associated complex Lie algebra. Then

(1) *$\mathfrak{g}(A)$ is a finite dimensional simple complex Lie algebra.*

(2) *The refined Coxeter-Dynkin diagram is one of the following:*



"simply laced" i.e. $a_{ij} \in \{0, -1\}$ if $i \neq j$ (A_r, D_r, E_6, E_7, E_8)

$a_{ij} \in \{0, -1, -2, -3\}$ if $i \neq j$ (G_2, F_4, B_r, C_r)

[17] Reminder: $h_j = [e_j, f_j]$

Proof. cf. [Knapp96] □

Exercise. For a semisimple finite dimensional complex Lie algebra $\mathfrak{g}$, the following statements are equivalent:

(1) $\mathfrak{g}$ is simple.

(2) The Cartan matrix is irreducible.

(3) The Coxeter-Dynkin diagram is connected.

Hint: If $\mathfrak{g} = \bigoplus_a \mathfrak{g}^{(a)}$ with $\mathfrak{g}^{(a)}$ simple for all a, what do the $\mathfrak{g}^{(a)}$ correspond to in terms of the Cartan matrix or the Coxeter-Dynkin diagram?

Theorem 2.3.10. (Denominator identity) *For any finite dimensional complex simple Lie algebra $\mathfrak{g}$ with notations as in Theorem 2.3.6, let*

$$\varrho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

denote the Weyl vector. *Then the following holds for all $z \in \mathfrak{h}$:*

$$\prod_{\alpha \in \Delta^+} \left(1 - \mathrm{e}^{-\alpha(z)}\right) = \sum_{\sigma \in W} \det(\sigma)\, \mathrm{e}^{\sigma(\varrho)(z) - \varrho(z)}$$

Remark. (Ideas for the proof of Theorem 2.3.10)
The denominator identity can be shown directly[18] [19]:

$$\begin{aligned} \mathrm{e}^{\varrho(z)} \prod_{\alpha \in \Delta^+} \left(1 - \mathrm{e}^{-\alpha(z)}\right) &= \prod_{\alpha \in \Delta^+} \left(\mathrm{e}^{\frac{\alpha(z)}{2}} - \mathrm{e}^{-\frac{\alpha(z)}{2}}\right) \\ &= \sum_{(\varepsilon_\alpha) \in \{\pm 1\}^{\Delta^+}} \left(\prod_{\alpha \in \Delta^+} \varepsilon_\alpha\right) \exp\left(\frac{1}{2} \underbrace{\left(\sum_{\alpha \in \Delta^+} \varepsilon_\alpha \alpha\right)}_{=:\delta} (z)\right) \end{aligned}$$

Via Theorem 2.3.6 (4), we see that $S_{\alpha_j}(\varrho) = \varrho - \alpha_j$ and $\det(S_{\alpha_j}) = -1 = \prod_\alpha \varepsilon_\alpha$. This already relates to $\det(\sigma)$ on the right hand side of the denominator identity.
If $\alpha, \beta, \gamma \in \Delta^+$ with $\alpha + \beta = \gamma$, then $(\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_\gamma) = \pm(+1, +1, -1)$ both give the same δ, but opposite $\prod_{\eta \in \Delta^+} \varepsilon_\eta$, leading to cancellations for the sum in the equation above.
Applying all these observations yields (after some computation):

$$\mathrm{e}^{\varrho(z)} \prod_{\alpha \in \Delta^+} \left(1 - \mathrm{e}^{-\alpha(z)}\right) = \sum_{\sigma \in W} \mathrm{e}^{\sigma(\varrho)(z)}$$

Exercise.

(1) With notations as above, show:

$$\forall\, j : \langle \varrho, \alpha_j \rangle = \frac{1}{2} \langle \alpha_j, \alpha_j \rangle$$

Also, setting $h_{\alpha_j} \in \mathfrak{h}$ such that $\varkappa(h_{\alpha_j}, X) = \alpha_j(X)$ for all $X \in \mathfrak{g}$ and

$$h_j := \frac{2 h_{\alpha_j}}{\alpha_j(h_{\alpha_j})}$$

as suggested by Theorem 2.3.6, we have $\varrho(h_j) = 1 \ \forall\, j$.

(2) Evaluate the denominator identity for $\mathfrak{sl}_n(\mathbb{C})$ and recover a formula for the Vandermonde determinant.

[18] With more machinery from representation theory (i.e. the Weyl character formula) it is just a corollary.
[19] There are some sign-related inconsistencies between us and some of Borcherds' publications.

2.4 Kac-Moody algebras

For this section, the main reference is [Kac90].

Definition 2.4.1.

(a) A Cartan-Kac-Moody (CKM) matrix is a matrix

$$A = (a_{ij}) \in \mathrm{Mat}_{\mathbb{Z}}(r \times r)$$

such that (C1) - (C3) from Theorem 2.3.6 hold, and

(C4′) There is a diagonal matrix[20] $D = \mathrm{diag}(d_1, \ldots, d_r), d_k > 0 \,\forall\, k$ such that

$$(DA)^\top = DA$$

Irreducibility is defined analogously to the irreducibility of Cartan matrices. Coxeter-Dynkin diagrams are also defined analogously, with the modification that the vertices i and j are connected by $\max\{|a_{ij}|, |a_{ji}|\}$ edges[21].

(b) If $A \in \mathrm{Mat}_{\mathbb{Z}}(r \times r)$ is a CKM matrix, then $\mathfrak{g}'(A)$ denotes the the abstract complex Lie algebra with generators $e_1, \ldots, e_r, f_1, \ldots, f_r, h_1, \ldots, h_r$ and relations (I) and (II) from Theorem 2.3.6.

Lemma 2.4.2. *Let $A \in \mathrm{Mat}_{\mathbb{Z}}(r \times r)$ denote a CKM matrix and assume that A is irreducible, then precisely one of the following holds*[22]*:*

(1) $\exists\, x > 0$ *such that* $Ax > 0$ *("finite type")*

(2) $\exists\, x > 0$ *such that* $Ax = 0$ *("affine type")*

(3) $\exists\, x > 0$ *such that* $Ax < 0$ *("indefinite" or "hyperbolic type")*

Furthermore:

(1) $\Longrightarrow$ $\det(A) \neq 0$ *and if* $Ax > 0$*, then* $x > 0$ *or* $x = 0$.

(2) $\Longrightarrow$ $\dim(\ker A) = 1$ *and if* $Ax \geq 0$*, then* $Ax = 0$.

(3) $\Longrightarrow$ *if* $Ax \geq 0$ *and* $x \geq 0$*, then* $x = 0$.

If $A = A^\top$, then (1) *is equivalent to A being positive definite and* (2) *is equivalent to A being positive semidefinite and* $\dim(\ker A) = 1$.

Theorem 2.4.3. *Let $A \in \mathrm{Mat}_{\mathbb{Z}}(r \times r)$ denote an irreducible CKM matrix, then*

(1) *If A is of finite type, then $\mathfrak{g}'(A)$ is finite dimensional and simple, and occurs in Serre's list (Theorem 2.3.9).*

(2) *If A is of affine type, then the Coxeter-Dynkin diagram is included in Table 2.4.1.*

Example 2.4.4.

(1) Let $\overline{\mathfrak{g}} = \mathfrak{sl}_2(\mathbb{C}) = \mathrm{span}_{\mathbb{C}}\{e_1, f_1, h_1\}$ with

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad h_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and let where $\varkappa$ denote the Killing form of $\overline{\mathfrak{g}}$. As an exercise, the following can be shown:

[20] That is, from (C4) we only drop the premiss of DA being positive definite.
[21] Notice, that by Theorem 2.3.9, every Cartan matrix obeys $a_{ij}a_{ji} = \max\{|a_{ij}|, |a_{ji}|\}$, however.
[22] For $x \in \mathbb{R}^d$: $x > 0 :\Longleftrightarrow x_k > 0 \;\forall\, k$

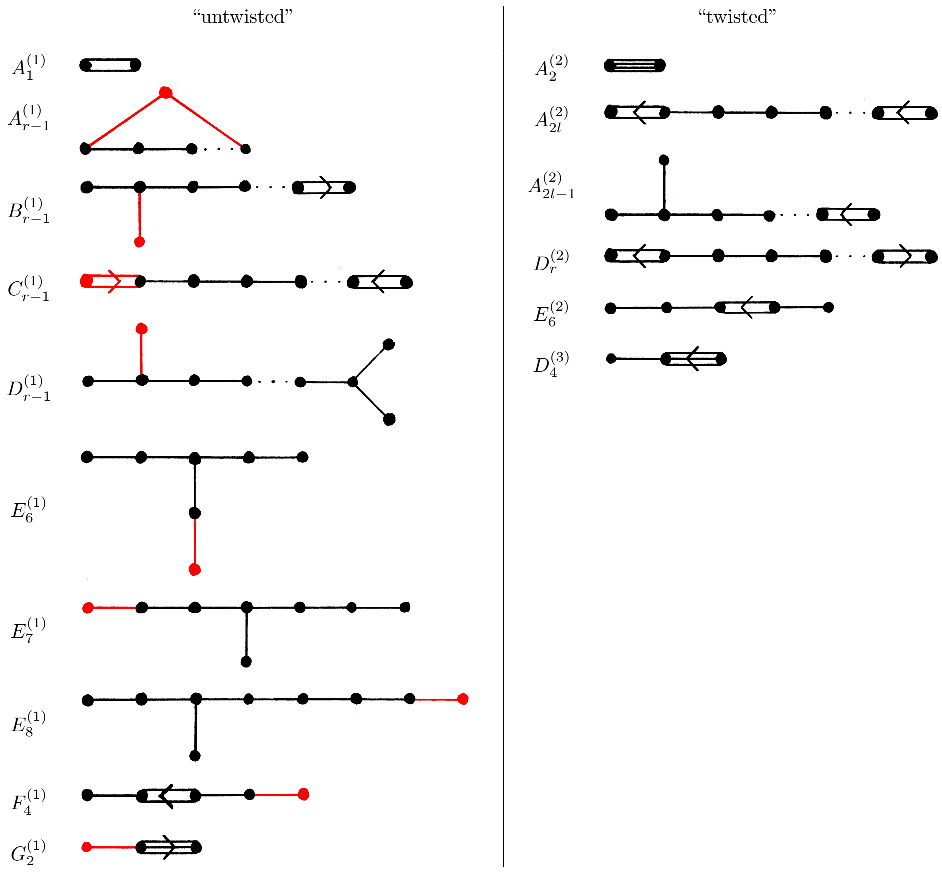


Table 2.4.1: Coxeter-Dynkin diagrams of affine CKM matrices. $(r = l + 1)$

(a) $\varkappa(h_1, h_1) = 8$, $\varkappa(e_1, f_1) = 4 = \varkappa(f_1, e_1)$ and for all other $X, Y \in \{e_1, f_1, h_1\}$ $\varkappa(X, Y) = 0$

(b) $\mathfrak{g}' := \bar{\mathfrak{g}} \otimes \mathbb{C}\left[t, t^{-1}\right] \oplus \mathbb{C}C$ with central $C \neq 0$ and $\forall\, X, Y \in \bar{\mathfrak{g}}, n, m \in \mathbb{Z}$:

$$[Xt^n, Yt^n] := [X, Y]\, t^{m+n} + \delta_{m+n,0}\varkappa(X, Y)\frac{C}{4}$$

yields an infinite dimensional Lie algebra[23].

(2) $\mathfrak{g}'$ is generated by e_1, f_1, h_1 along with

$$e_0 := \begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix}, \quad f_0 := \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \quad \text{and} \quad h_0 := C - h_1.$$

Indeed:

$$\begin{aligned} [e_1, e_0] &= t^{-1}h_1, & \left[t^{-1}h_1, t^n e_1\right] &= t^{n-1}2e_1 & \forall\, n \in \mathbb{Z} \\ [f_1, f_0] &= -th_1, & [th_1, t^n e_1] &= t^{n+1}2e_1 & \forall\, n \in \mathbb{Z} \\ \left[t^{-1}h_1, t^n f_1\right] &= -t^{n-1}2f_1, & [th_1, t^n f_1] &= -t^{n+1}2f_1 & \forall\, n \in \mathbb{Z} \end{aligned}$$

[23] The Lie algebra $\bar{\mathfrak{g}} \otimes \mathbb{C}\left[t, t^{-1}\right]$ (with the obvious Lie bracket) is called loop algebra. So basically, $\mathfrak{g}'$ is a central extension of a loop algebra.

Now, we look for $A \in \mathrm{Mat}_{\mathbb{Z}}(2 \times 2)$ such that $\mathfrak{g}' = \mathfrak{g}'(A)$. *22 Nov 2017*
First, we inspect the relations belonging to (I): For all $i, j \in \{0, 1\}$, we have

$$[h_i, h_j] = 0 \quad \text{and} \quad [e_i, f_j] = \delta_{ij} h_j$$

where for the latter equality we use $[e_0, f_0] = [f_1, e_1] + \varkappa(f_1, e_0)\frac{C}{4} = h_0$. Moreover,

$$\begin{aligned} [h_1, e_1] &= \ \ 2e_1, \quad [h_1, e_0] = -2e_0 \\ [h_0, e_1] &= -2e_1, \quad [h_0, e_0] = \ \ 2e_0 \end{aligned}$$

and thus $A := \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ yields $[h_i, e_j] = a_{ij} e_j$. One can also check that

$$[h_i, f_j] = -a_{ij} f_j$$

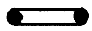
Concerning (II), we see

$$\mathrm{ad}_{e_1}^{1-a_{10}}(e_0) = [e_1, \underbrace{[e_1, \underbrace{[e_1, e_0]}_{t^{-1}h_1}]}_{-2t^{-1}e_1}] = 0$$

Analogously, all of (II) can be validated:

$$\forall\, i, j \in \{0, 1\} \text{ with } i \neq j : \mathrm{ad}_{e_i}^{1-a_{ij}}(e_j) = \mathrm{ad}_{f_i}^{1-a_{ij}}(f_j) = 0$$

Thus, we established $\mathfrak{g}' = \mathfrak{g}'(A)$ with the Coxeter-Dynkin diagram being $A_1^{(1)}$:



(3) The other $Z_m^{(1)}, Z \in \{A, \dots, G\}$ are constructed analogously from the simple finite dimensional Lie algebras of type Z_m.
Then, the $Z_m^{(2,3)}$ arise from "twisting" some $Z_m^{(1)}$ by $C = \mathbb{Z}_2, \mathbb{Z}_3$, an automorphism group of $Z_m^{(1)}$, i.e. $Z_m^{(2,3)} = Z_m^{(1)}/C$.

Remark 2.4.5. For $\overline{\mathfrak{g}} = \mathfrak{sl}_2(\mathbb{C})$ with notations as in Example 2.4.4 (e.g. $\mathfrak{g}' = \mathfrak{g}'(A)$), we run into the problem that the simple roots are linearly dependent [24]:
While h_0, h_1 form a basis of $\mathfrak{h}'$,

$$\mathfrak{g}' \supset \mathfrak{h}' = \mathrm{span}_{\mathbb{C}}\{h_0, h_1\} \cong \mathbb{C}^2$$

we have $\mathrm{ad}_{h_i}(e_j) = \alpha_j(h_i)\, e_j$. Thus, with respect to the basis (h_0, h_1) of $\mathfrak{h}'$, we get the following matrix representations[25]:

$$\alpha_0 \mathrel{\widehat{=}} \begin{pmatrix} 2 & -2 \end{pmatrix} \quad \text{and} \quad \alpha_1 \mathrel{\widehat{=}} \begin{pmatrix} -2 & 2 \end{pmatrix}$$

These two row vectors are obviously linearly dependent.

As a remedy, we consider instead of $\mathfrak{g}'$

$$\mathfrak{g} := \mathfrak{g}' \oplus \mathbb{C} l_0,$$

[24] This leads to a problem in the representation theory of $\mathfrak{g}'$: The weight spaces are not finite dimensional. [Gannon98]
[25] These are just the rows of the CKM matrix, so the linear dependence of the α_j was to be expected for the affine case, where we required A to be non-invertible.

$$[l_0, C] = 0 \quad \text{and} \quad [l_0, t^n X] = -n\, t^n X \quad \forall\, n \in \mathbb{Z}, X \in \overline{\mathfrak{g}}$$

Then $\mathfrak{h} = \operatorname{span}_{\mathbb{C}} \{h_0, h_1, l_0\}$ and α_0, α_1 have to be linearly extended:

$$\begin{aligned} &[l_0, e_0] = e_0, \ [l_0, f_0] = -f_0 &&\implies \alpha_0(l_0) = 1 \\ &[l_0, e_1] = 0, \ \ [l_0, f_1] = 0 &&\implies \alpha_1(l_0) = 0 \end{aligned}$$

So now, with respect to the basis (h_0, h_1, l_0) of $\mathfrak{h}$, we obtain as matrix representations:

$$\alpha_0 \mathrel{\widehat{=}} \begin{pmatrix} 2 & -2 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_1 \mathrel{\widehat{=}} \begin{pmatrix} -2 & 2 & 0 \end{pmatrix}$$

Hence, we "made" the α_j linearly independent.

Exercise. Show that the Witt algebra from Example 2.1.5 acts on $\mathfrak{g}'$ via $\varrho(l_n) := -t^{1-n}\partial_t$.

l_0 is called a DERIVATION. In general (for any Lie algebra $\mathfrak{g}'$), a derivation is a vectorspace endomorphism $\delta : \mathfrak{g}' \to \mathfrak{g}'$ such that

$$\delta([X, Y]) = [\delta(X), Y] + [X, \delta(Y)] \tag{$*$}$$

"Adjoining" such a derivation to $\mathfrak{g}'$ means taking the direct sum

$$\mathfrak{g} := \mathfrak{g}' \oplus \mathbb{C} l_0$$

where $\operatorname{ad}_{l_0} = \delta$, so

$$\forall\, X \in \mathfrak{g}' : [l_0, X] = \delta(X)$$

As an exercise it can be shown that the condition $(*)$ ensures that the above defines indeed a Lie algebra.

Definition 2.4.6. Let $r > 0$, $A = (a_{ij})_{i,j \in \{0,\dots,r-1\}} \in \operatorname{Mat}_{\mathbb{Z}}(r \times r)$ a CKM matrix. Denote by $\mathfrak{g}' = \mathfrak{g}'(A)$ the associated complex Lie algebra from Definition 2.4.1. Define

$$\mathfrak{h}' := \operatorname{span}_{\mathbb{C}} \{h_0, \dots, h_{r-1}\}$$

and for $j \in \{0, \dots, r-1\}$ let $\alpha_j \in (\mathfrak{h}')^*$ such that for all $H \in \mathfrak{h}'$

$$\begin{aligned} [H, e_j] &= \ \ \alpha_j(H)\, e_j \\ [H, f_j] &= -\alpha_j(H)\, f_j \end{aligned}$$

Let $\mathscr{D}$ denote a complex vectorspace of minimal dimension such that a linear extension of the α_j to the Abelian Lie algebra

$$\mathfrak{h} := \mathfrak{h}' \oplus \mathscr{D}$$

exists (still denoted α_j) with $(\alpha_0, \dots, \alpha_{r-1})$ linearly independent. Then define:

(1) $\mathfrak{g}(A) := \mathfrak{g}'(A) \oplus \mathscr{D}$ is called the KAC-MOODY ALGEBRA associated to A, where $\mathfrak{h}$ is Abelian and for all $D \in \mathscr{D}, j \in \{0, \dots, r-1\}$

$$\begin{aligned} [D, e_j] &= \ \ \alpha_j(D)\, e_j \\ [D, f_j] &= -\alpha_j(D)\, f_j \end{aligned}$$

(2) For all $\alpha \in \mathfrak{h}^*$ let

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}(A) \mid \operatorname{ad}_H(X) = \alpha(H) X \ \ \forall\, H \in \mathfrak{h}\}$$

Then define the set

$$\Delta := \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \, \dim \mathfrak{g}_\alpha \neq 0\}$$

and call the elements of Δ ROOTS. The α_j are called SIMPLE ROOTS.

(3) For $j \in \{0, \dots, r-1\}$, let

$$S_{\alpha_j} : \mathfrak{h}^* \to \mathfrak{h}^* : \xi \mapsto \xi - 2\frac{\xi(h_j)}{\alpha_j(h_j)}\alpha_j = \xi - \xi(h_j)\alpha_j$$

Then the group $W := \langle S_{\alpha_j} \mid j \in \{0, \dots, r-1\} \rangle$ is called the WEYL GROUP of $\mathfrak{g}(A)$.

(4) If for $\alpha \in \Delta$ we have $\sigma(\alpha_j) = \alpha$ for some $\sigma \in W$, $j \in \{1, \dots, r-1\}$, then α is called a REAL ROOT. Otherwise, α is IMAGINARY.

Exercise 2.4.7. With notations as in Remark 2.4.5

$$\begin{aligned} \overline{\mathfrak{g}} &= \mathfrak{sl}_2(\mathbb{C}) = \operatorname{span}_{\mathbb{C}} \{e_1, f_1, h_1\} \\ \overline{\mathfrak{h}} &= \operatorname{span}_{\mathbb{C}} \{h_1\} \end{aligned}$$

$\overline{\Delta}$ the root system of $\overline{\mathfrak{g}}$, $\mathfrak{g}$ the Kac-Moody algebra associated to

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

(1) Show that

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{n \in \mathbb{Z}} \bigoplus_{\overline{\alpha} \in \overline{\Delta}} t^n \overline{\mathfrak{g}}_{\overline{\alpha}} \oplus \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} t^n \overline{\mathfrak{h}}$$

where $\mathfrak{h} = \operatorname{span}_{\mathbb{C}} \{h_0, h_1, l_0\}$ is the Cartan subalgebra. The root system of $\mathfrak{g}$ is

$$\Delta = \{\pm\alpha_1 + n\delta \mid n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z} \setminus \{0\}\}$$

with $\delta := \alpha_0 + \alpha_1$. Moreover, $\mathfrak{g}$ admits a decomposition

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_- \\ \text{where} \quad \mathfrak{g}_\pm &= \mathbb{C}[t^{\mp 1}] \otimes \overline{\mathfrak{g}}_\pm \ \oplus \ t^{\mp 1}\mathbb{C}[t^{\mp 1}] \otimes \left(\overline{\mathfrak{g}}_\mp \oplus \overline{\mathfrak{h}}\right) \end{aligned}$$

and $\mathfrak{g}_+, \mathfrak{g}_-$ are generated by the e_j, f_j, respectively.

(2) Generalize this to all $A_{r-1}^{(1)}$.

(3) For $A_1^{(1)}$, calculate $S_{\alpha_0}, S_{\alpha_1}$ and show that $\{\pm\alpha_1 + n\delta \mid n \in \mathbb{Z}\}$ are the real roots and $\{n\delta \mid n \in \mathbb{Z} \setminus \{0\}\}$ are the imaginary ones.

Theorem 2.4.8. *Let $A \in \operatorname{Mat}_{\mathbb{Z}}(r \times r)$ denote a CKM matrix, and recall the notations of Definition 2.4.6.*

(1) $\mathfrak{g} := \mathfrak{g}(A)$ *as in Definition 2.4.6 exists and is unique up to isomorphism.*

(2) *The following are equivalent:*

  (a) *A is of finite type.*

  (b) $\dim \mathfrak{g} < \infty$

  (c) $\#\Delta < \infty$

  (d) $\#W < \infty$

(3) $\mathfrak{h} \subset \mathfrak{g}$ *is a Cartan subalgebra and* $\mathfrak{g}$ *has a root space decomposition with respect to* $\operatorname{ad}_{\mathfrak{h}}$ *such that the following holds:*

  (a) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$ *with* $\mathfrak{g}_+, \mathfrak{g}_-$ *generated by the* e_j, f_j*, respectively.*

(b) *The following statements from the semisimple case continue to hold:*

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \overset{\mathfrak{h}=\mathfrak{g}_0}{=} \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta}$$

$$\Delta = \Delta^+ \cup \Delta^-, \quad \textit{where} \quad \Delta^- = -\Delta^+ \quad \textit{and} \quad \Delta^+ \subset \operatorname{span}_{\mathbb{N}} \{\alpha_0, \dots, \alpha_{r-1}\}$$

Furthermore, we have $\mathfrak{g}_\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$ *and* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ *for all* $\alpha, \beta \in \mathfrak{h}^*$.

(c) *For all* $\alpha \in \mathfrak{h}^*$, $\dim \mathfrak{g}_\alpha < \infty$ *holds. If* α *is a real root, then* $\dim \mathfrak{g}_\alpha = 1$ *and* $n\alpha \in \Delta$ *implies that* $n \in \{\pm 1\}$.

(d) *There is an invariant symmetric bilinear form* $\langle \cdot, \cdot \rangle$ *on* $\mathfrak{g}$ *such that for all* $\alpha \in \Delta$, *the restriction* $\langle \cdot, \cdot \rangle|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ *is non-degenerate, and if* $\beta \in \mathfrak{h}^* \setminus \{-\alpha\}$, *then*

$$\langle \cdot, \cdot \rangle|_{\mathfrak{g}_\alpha \times \mathfrak{g}_\beta} = 0$$

There is a linear map

$$\operatorname{span}_{\mathbb{N}} \Delta \to \mathfrak{h} : \alpha \mapsto h_\alpha$$

such that for all $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$

$$[X, Y] = \langle X, Y \rangle h_\alpha, \quad \langle h_\alpha, h_\beta \rangle = \alpha(h_\beta) = \beta(h_\alpha)$$

(For all $\alpha, \beta \in \operatorname{span}_{\mathbb{C}} \Delta \subset \mathfrak{h}^*$, *we define* $\langle \alpha, \beta \rangle := \langle h_\alpha, h_\beta \rangle$

(e) $\alpha \in \Delta$ *is real iff* $\langle \alpha, \alpha \rangle > 0$.

(f) *There is a* $\mathbb{C}$*-antilinear continuation* ϑ *of*

$$e_j \leftrightarrow -f_j, \; h_j \mapsto -h_j \quad \forall\, j \in \{1, \dots, r-1\}$$

to a Lie algebra homomorphism (over $\mathbb{R}$*) such that* $\vartheta^2 = \mathrm{id}$, $\vartheta|_{\mathfrak{h}} = -\mathrm{id}$ *and*

$$(X, Y) := -\langle X, \vartheta Y \rangle \quad \forall\, X, Y \in \mathfrak{g}$$

is positive definite on each $\mathfrak{g}_\alpha$, $\alpha \in \Delta$.

27 Nov 2017 **Theorem 2.4.9.** *Let* $r \in \mathbb{N} \setminus \{0\}$, $A = (a_{ij})_{i,j \in \{0,\dots,r-1\}} \in \operatorname{Mat}_{\mathbb{Z}}(r \times r)$ *a CKM matrix. Let* $\mathfrak{g}'_{\mathrm{KM}}(A)$ *denote the* real *Lie algebra with generators*

$$e_0, \dots, e_{r-1}, f_0, \dots, f_{r-1}, h_0, \dots, h_{r-1}$$

and relations (I) *and* (II) *from Theorem 2.3.6. Let*

$$\mathfrak{h}'_{\mathbb{R}} := \operatorname{span}_{\mathbb{R}} \{h_0, \dots, h_{r-1}\}$$

and $\alpha_0, \dots, \alpha_{r-1} \in (\mathfrak{h}'_{\mathbb{R}})^*$ *with*

$$\forall i, j : [h_i, e_j] = \underbrace{\alpha_j(h_i)}_{= a_{ij} \in \mathbb{Z}} e_j, \quad [h_i, f_j] = -\alpha_j(h_i) f_j$$

Let $\mathscr{D}'_{\mathbb{R}}$ *denote a space of derivations (*$\mathscr{D}'_{\mathbb{R}} \subset \operatorname{End}_{\mathbb{R}} \mathfrak{g}'_{\mathrm{KM}}$*) of minimal dimension such that there are linearly independent linear extensions of the* α_j *to the Abelian Lie algebra* $\mathfrak{h}_{\mathbb{R}} := \mathfrak{h}'_{\mathbb{R}} \oplus \mathscr{D}'_{\mathbb{R}}$ *(still denoted by* α_j*). Let*

$$\mathfrak{g}_{\mathrm{KM}}(A) := \mathfrak{g}'_{\mathrm{KM}}(A) \oplus \mathscr{D}'_{\mathbb{R}}$$

denote the Lie algebra obtained by adjoining these derivations according to $\alpha_0, \dots, \alpha_{r-1}$.

(1) $\mathfrak{g}_{\mathrm{KM}}(A)$ *is a real form*[26] *of the Kac-Moody algebra* $\mathfrak{g}(A)$ *from Definition 2.4.6.*

[26] i.e. $\mathfrak{g}(A) = \mathbb{C} \otimes \mathfrak{g}_{\mathrm{KM}}(A)$

(2) *The properties (3) (a) - (e) of Theorem 2.4.8 hold analogously for* $\mathfrak{g}_{\mathrm{KM}}(A)$*:*

- $\mathfrak{g}_{\mathrm{KM}}(A) = \underbrace{\mathfrak{h}_{\mathbb{R}}}_{\mathfrak{g}_0^{\mathbb{R}}} \oplus \underbrace{\bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha^{\mathbb{R}}}_{\text{gen. by the } e_j} \oplus \underbrace{\bigoplus_{\alpha\in\Delta^-} \mathfrak{g}_\alpha^{\mathbb{R}}}_{\text{gen. by the } f_j}$
- $\forall\, \alpha, \beta \in \mathfrak{h}_{\mathbb{R}}^* : [\mathfrak{g}_\alpha^{\mathbb{R}}, \mathfrak{g}_\beta^{\mathbb{R}}] \subset \mathfrak{g}_{\alpha+\beta}^{\mathbb{R}}$, *and* $\dim \mathfrak{g}_\alpha^{\mathbb{R}} < \infty$ *if* $\alpha \neq 0$
- *The root system can be decomposed into* $\Delta = \Delta^+ \cup \Delta^-$ *with* $\Delta^- = -\Delta^+$ *and* $\Delta^+ \subset \mathrm{span}_{\mathbb{N}} \{\alpha_0, \ldots, \alpha_{r-1}\}$.
- *There is an invariant symmetric bilinear form* $\langle\,\cdot\,,\,\cdot\,\rangle$ *on* $\mathfrak{g}_{\mathrm{KM}}(A)$ *such that*

$$\forall\, \alpha, \beta \in \Delta : \langle\,\cdot\,,\,\cdot\,\rangle\,|_{\mathfrak{g}_\alpha^{\mathbb{R}} \times \mathfrak{g}_\beta^{\mathbb{R}}} = \begin{cases} 0, & \text{if } \alpha+\beta \neq 0 \\ \text{non-degenerate}, & \text{else} \end{cases}$$

$$\forall\, X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_{-\alpha} : [X, Y] = \langle X, Y\rangle\, h_\alpha$$

  where $\mathrm{span}_{\mathbb{R}}\Delta \to \mathfrak{h}_{\mathbb{R}} : \alpha \mapsto h_\alpha$ *is a linear map. Moreover,*

$$\forall\, \alpha, \beta \in \mathrm{span}_{\mathbb{R}}\Delta : \langle h_\alpha, h_\beta\rangle = \alpha(h_\beta) = \beta(h_\alpha) =: \langle \alpha, \beta\rangle$$

- *If* $D = \mathrm{diag}(d_0, \ldots, d_{r-1})$ *with* $d_j > 0$ *and* $(DA)^\top = DA$*, then*

$$h_{\alpha_j} := d_j h_j \quad \text{and} \quad \langle \alpha_i, \alpha_j\rangle = d_i a_{ij} \quad \forall\, i, j$$

- *There is a "Cartan[27] involution"* ϑ*, that is,* ϑ *is a Lie algebra homomorphism with* $\vartheta^2 = \mathrm{id}, \vartheta : e_j \leftrightarrow -f_j, h_j \mapsto -h_j$ *and* $\vartheta|_{\mathfrak{g}_0} = -\mathrm{id}_{\mathfrak{g}_0}$ *which leaves* $\langle\,\cdot\,,\,\cdot\,\rangle$ *invariant and yields* $(X, Y) \mapsto -\langle X, \vartheta Y\rangle$*, a bilinear form that is positive definite on* $\mathfrak{g}_\alpha$ *for all* $\alpha \in \Delta$.

(3) W *preserves* $\langle\,\cdot\,,\,\cdot\,\rangle$ *and* $\alpha \in \Delta$ *is a real root iff* $\langle \alpha, \alpha\rangle > 0$*. For every real root* α*, we have* $\dim \mathfrak{g}_\alpha = 1$.

(4) $(\,\cdot\,,\,\cdot\,) := -\langle\,\cdot\,, \vartheta(\,\cdot\,)\rangle$ *is positive definite on all of* $\mathfrak{g}_{\mathrm{KM}}(A)$ *if and only if* $\mathfrak{g}_{\mathrm{KM}}(A)$ *is a finite dimensional simple Lie algebra.*

Remark. The definition of Kac-Moody algebras is not uniform in the literature:

- One can work on any field of characteristic zero.
- One can omit the adjoining of derivatives or divide by central Lie subalgebras.
- One can omit (C4)' (Kac does that! That's why we listed (C3) separately.)

Then one needs to make additional assumptions to obtain some of Theorem 2.4.9.

Exercise 2.4.10.

(1) With notations as in Example 2.4.4, let $\mathfrak{g}$ denote the Kac-Moody algebra of type $A_1^{(1)}$. Check that $\alpha \in \Delta$ is real iff $\langle \alpha, \alpha\rangle > 0$.

(2) If $\overline{\mathfrak{g}}$ is a finite dimensional complex Lie algebra, let

$$\widehat{\mathfrak{g}} := \mathbb{C}[t, t^{-1}] \otimes \overline{\mathfrak{g}} \quad \text{with} \quad [t^n X, t^m Y] = t^{m+n}\, [X, Y]$$

denote the (POLYNOMIAL) LOOP ALGEBRA[28] associated to $\overline{\mathfrak{g}}$. Show that if $\overline{\mathfrak{g}}$ is semisimple, so is $\widehat{\mathfrak{g}}$.

[27] The literature is not entirely consistent in calling ϑ a Cartan involution.
[28] The name comes from considering $t \in S^1 \subset \mathbb{C}$

Theorem 2.4.11. (Denominator Identity) *For any irreducible CKM matrix A consider the associated Kac-Moody algebra $\mathfrak{g}$. With notations as in Theorem 2.4.8, the following holds:*
If $\varrho \in \mathfrak{h}^*$ obeys $\langle \varrho, \alpha_j \rangle = \frac{1}{2} \langle \alpha_j, \alpha_j \rangle$ (equivalent: $\varrho(h_j) = 1$) for all j, then:

$$\prod_{\alpha \in \Delta^+} \left(1 - \mathrm{e}^{-\alpha}\right)^{\dim \mathfrak{g}_\alpha} = \mathrm{e}^{-\varrho} \sum_{\sigma \in W} \det(\sigma) \mathrm{e}^{\sigma(\varrho)}$$

Remark.

- ϱ as in Theorem 2.4.11 is called a Weyl vector of a Kac-Moody algebra. It is unique iff A is of finite type.
- Here and in the following, for $\beta \in \mathfrak{h}^*$

$$\mathrm{e}^\beta : \mathfrak{h} \to \mathbb{C} : z \mapsto \mathrm{e}^{\beta(z)}$$

  The formula in Theorem 2.4.11 must be read as an identity between functions on the domain where both sides converge.

Exercise. For $\mathfrak{g}$ of type $A_1^{(1)}$, show that the denominator identity yields the Jacobi triple product identity: Insert $z = xh_0 - 2tl_0$, $r := \mathrm{e}^{2x}$, $q := \mathrm{e}^{2t}$ to obtain

$$\prod_{n=0}^{\infty} (1 - rq^n) \prod_{n=1}^{\infty} \left(1 - r^{-1}q^n\right)(1 - q^n) = \sum_{m=-\infty}^{\infty} (-1)^m r^{-m} q^{\frac{m}{2}(m+1)}$$

which converges if $x \in \mathbb{C}$, $t \in \mathbb{C}$ with $\mathrm{Re}(t) < 0$ (or, equivalently, $t = \imath\tau$ with $\tau \in \mathfrak{H}$).

2.5 Borcherds-Kac-Moody Algebras

Borcherds, Jurisich, Ray	Gannon
generalized Kac-Moody algebra	Borcherds-Kac-Moody algebra
symmetrized generalized Kac-Moody matrix	Borcherds-Kac-Moody matrix

Table 2.5.1: The terminology varies in the literature.

In this section, we will define Borcherds-Kac-Moody algebras. Note that this definition (as well as the terminology) is far from uniform in the literature (Table 2.5.1). For this subsection (apart from [Gannon98]) the following references are relevant: [Borcherds88], [Jurisich96] and [Ray2000].

Definition 2.5.1. Let I denote a finite index set or $I = \mathbb{N}$. Then

$$A = (a_{ij})_{i,j\in I} \in \mathbb{R}^{I\times I}$$

is called Borcherds-Kac-Moody matrix iff $\forall\, i, j \in I$

(BC1) $a_{ii} = 2$ or $a_{ii} \leq 0$

(BC2) $(i \neq j \implies a_{ij} \leq 0)$ and $(a_{ii} = 2 \implies a_{ij} \in \mathbb{Z})$

(BC3) $\exists\, (d_j)_{j\in I}$ with $d_j > 0\ \forall\, j$ such that $D := \mathrm{diag}(d_j)_{j\in I}$ satisfies[29]

$$(DA)^\top = DA$$

Irreducibility of A is defined as for the Cartan matrices. The universal Borcherds-Kac-Moody algebra $\widehat{\mathfrak{g}} := \widehat{\mathfrak{g}}_{\mathrm{BKM}}(A)$ is the real Lie algebra generated by e_j, f_j, h_{ij}, $i, j \in I$ and relations

(BI) $\forall\, i, j, k \in I$:

$$\begin{aligned} [e_j, f_k] &= h_{jk} \\ [h_{ij}, e_k] &= \delta_{ij} a_{ik} e_k \\ [h_{ij}, f_k] &= -\delta_{ij} a_{ik} e_k \end{aligned}$$

(BII) $\forall\, i, j \in I$ with $a_{ii} = 2$:

$$\begin{aligned} (\mathrm{ad}_{e_i})^{1-a_{ij}}(e_j) &= 0 \\ (\mathrm{ad}_{f_i})^{1-a_{ij}}(f_j) &= 0 \end{aligned}$$

(BIII) $\forall\, i, j \in I$ with $a_{ij} = 0 : [e_i, e_j] = [f_i, f_j] = 0$

Imposing, additionally, $h_{ij} = 0$ if $i \neq j$, the resulting Lie algebra is a Borcherds-Kac-Moody algebra[30] $\mathfrak{g}'_{\mathrm{BKM}}(A)$. *29 Nov 2017*

Remark 2.5.2. (exercises) Let A denote a Borcherds-Kac-Moody matrix and consider $\widehat{\mathfrak{g}}_{\mathrm{BKM}}(A)$, $\mathfrak{g}'_{\mathrm{BKM}}(A)$ as in Definition 2.5.1. Show:

(0) From (BC1) - (BC3) follows

$$\forall\, i, j \in I : a_{ij} = 0 \implies a_{ji} = 0$$

[29] This property is important to get the bilinear form.

[30] Keep in mind, this is the definition of $\mathfrak{g}'_{\mathrm{BKM}}(A)$ and *not* of Borcherds-Kac-Moody algebras in general.

(1) In $\widehat{\mathfrak{g}}_{\mathrm{BKM}}(A)$, the following holds:

   (a) $\forall\, i,j,k,l \in I : [h_{ij}, h_{kl}] = 0$

   (b) If $h_{ij} \neq 0$, then the i^{th} and the j^{th} column of A agree.

(2) $\widehat{\mathfrak{g}}_{\mathrm{BKM}}(A)$ is a central extension of $\mathfrak{g}'_{\mathrm{BKM}}(A)$.

(3) $\forall\, j \in I$:

   (a) If $a_{jj} \neq 0$, then $\mathrm{span}_{\mathbb{C}}\{e_j, f_j, h_{jj}\} \cong \mathfrak{sl}_2(\mathbb{C})$.

   (b) If $a_{jj} = 0$, then $\mathrm{span}_{\mathbb{C}}\{e_j, f_j, h_{jj}\}$ is isomorphic to the HEISENBERG ALGEBRA[31] $\mathrm{span}_{\mathbb{C}}\{p, q, \mathbb{1}\}$ with central $\mathbb{1}$ and $[p,q] = -\imath\hbar\mathbb{1}$.

(4) (not an exercise) For any vectorspace V over $\mathbb{C}$, a real vectorspace $V_{\mathbb{R}}$ is called REAL FORM if $V = V_{\mathbb{R}} \oplus \imath V_{\mathbb{R}}$.
For a complex Lie algebra $\mathfrak{g}$, a real form is a Lie algebra $\mathfrak{g}_{\mathbb{R}}$ over $\mathbb{R}$ such that $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \oplus \imath\mathfrak{g}_{\mathbb{R}}$.
Call $\mathfrak{g}_{\mathbb{R}}$ a COMPACT REAL FORM if it is also pointwise fixed by some Cartan involution.

Definition 2.5.3. Let A denote a Borcherds-Kac-Moody matrix and $\mathfrak{g}'_{\mathrm{BKM}}(A)$ as in Definition 2.5.1.
Let $\mathfrak{h}' := \mathrm{span}_{\mathbb{R}}\{h_{jj} \mid j \in I\} \cong \widehat{\mathfrak{h}}/\langle h_{ij} \mid i \neq j\rangle$, where $\widehat{\mathfrak{h}} := \mathrm{span}_{\mathbb{R}}\{h_{ij} \mid i,j \in I\} \subset \widehat{\mathfrak{g}}_{\mathrm{BKM}}(A)$. For all $j \in I$, define $\alpha_j \in (\mathfrak{h}')^*$ such that

$$\forall\, H \in \mathfrak{h}' : \begin{cases} \mathrm{ad}_H(e_j) &= \alpha_j(H) e_j \\ \mathrm{ad}_H(f_j) &= -\alpha_j(H) f_j \end{cases} \qquad (*)$$

and let $\mathscr{D}$ denote a minimal space of pairwise commuting derivations (on $\mathfrak{g}'_{\mathrm{BKM}}(A)$) such that there exists a choice of linear extensions of the α_j (still denoted α_j) to the Abelian Lie algebra $\mathfrak{h}' \oplus \mathscr{D}$ which are linearly independent. Then define

(1) $\mathfrak{g}_{\mathrm{BKM}}(A) := \mathfrak{g}'_{\mathrm{BKM}}(A) \oplus \mathscr{D}$ with Abelian $\mathfrak{h} := \mathfrak{h}' \oplus \mathscr{D}$ and by extending $(*)$ to all $H \in \mathfrak{h}$.

(2) $\forall\, \alpha \in \mathfrak{h}^*$:
$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}_{\mathrm{BKM}}(A) \mid \forall\, H \in \mathfrak{h} : \mathrm{ad}_H(X) = \alpha(H)X\}$$
The ROOT SYSTEM OF $\mathfrak{g}_{\mathrm{BKM}}(A)$ is
$$\Delta := \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq \{0\}\}$$
and $\alpha_j \in \Delta$ are the SIMPLE ROOTS. We call simple roots α_j REAL if $a_{jj} = 2$, otherwise, α_j is IMAGINARY.

(3) Given $j \in I$ such that α_j is real, $S_{\alpha_j} : \mathfrak{h}^* \to \mathfrak{h}^*$ is defined as before in Definition 2.4.6.
$$W := \left\langle S_{\alpha_j} \,\middle|\, j \in I \quad \text{with} \quad a_{jj} = 2 \right\rangle$$
is the WEYL GROUP OF $\mathfrak{g}_{\mathrm{BKM}}(A)$.

(4) $\alpha \in \Delta$ is a REAL ROOT iff $\exists\, \sigma \in W, j \in I$ with $a_{jj} = 2$ such that $\sigma(\alpha_j) = \alpha$. If $\alpha \in \Delta$ is not real, then it is an IMAGINARY ROOT.

(5) $\Lambda := \mathrm{span}_{\mathbb{Z}}(\Delta)$ is called the ROOT LATTICE OF $\mathfrak{g}_{\mathrm{BKM}}(A)$.

Theorem 2.5.4. [Borcherds88] *If A is a Borcherds-Kac-Moody matrix and $\mathfrak{g} = \mathfrak{g}_{\mathrm{BKM}}(A)$ from Definition 2.5.3, then all the properties (2), (3) from Theorem 2.4.9 hold for $\mathfrak{g}$.*

[31] If (V,ω) is a symplectic vectorspace, then it gives rise to a Lie algebra: the associated Heisenberg algebra $V \oplus \mathbb{C}\mathbb{1}$ with $\forall\, v,w \in V : [v,w] = \omega(v,w)\mathbb{1}$ and $\mathbb{1}$ central. (from Guillemin/Sternberg 'Geometric Asymptotics' (1977) p.276)

Remark. The root space decomposition

$$\mathfrak{g}_{\mathrm{BKM}}(A) = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathfrak{g}_\alpha$$

imposes a GRADING on $\mathfrak{g}_{\mathrm{BKM}}(A)$ by the root lattice $\Lambda = \mathrm{span}_{\mathbb{Z}} \{\alpha_j \mid j \in I\}$, i.e. any $X \in \mathfrak{g}_\alpha$, $(\alpha \in \Delta \cup \{0\})$ has degree α and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}\ \forall\, \alpha, \beta \in \Lambda$.
By choosing any $\mathbb{Z}$-module homomorphism $\Lambda \to \mathbb{Z}$, this induces a $\mathbb{Z}$-grading:

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n \quad \text{and} \quad \forall\, m, n \in \mathbb{Z} : [\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{m+n}$$

Definition 2.5.5. [Borcherds '91] A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is called a BORCHERDS-KAC-MOODY ALGEBRA iff $\mathfrak{g}$ is isomorphic to a Lie algebra obtained from a universal Borcherds-Kac-Moody algebra $\widehat{\mathfrak{g}}_{\mathrm{BKM}}(A)$ by quotienting by a central Lie subalgebra of $\widehat{\mathfrak{h}}$ and adjoining some Abelian Lie algebra of derivations.

In other words: $\mathfrak{g} = \mathfrak{g}' \oplus \mathscr{D}$, where $\mathscr{D} \subset \mathrm{End}_{\mathbb{R}}(\mathfrak{g}')$ is an Abelian Lie algebra of derivations and there is a short exact sequence of Lie algebras

$$0 \to \mathfrak{a} \overset{\iota}{\hookrightarrow} \widehat{\mathfrak{g}}_{\mathrm{BKM}}(A) \overset{\pi}{\twoheadrightarrow} \mathfrak{g}' \to 0$$

with central $\iota(\mathfrak{a}) \subset \widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}_{\mathrm{BKM}}(A)$.
In particular, all of $\widehat{\mathfrak{g}}_{\mathrm{BKM}}(A), \mathfrak{g}'_{\mathrm{BKM}}(A), \mathfrak{g}_{\mathrm{BKM}}(A)$ with Borcherds-Kac-Moody matrix A are Borcherds-Kac-Moody algebras.

Theorem 2.5.7. [Borcherds '92] *Let $\mathfrak{g}$ denote a real Lie algebra with the following properties:*

(a) $\mathfrak{g}$ *is $\mathbb{Z}$-graded, i.e.*

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n \quad \textit{and} \quad \forall\, m, n \in \mathbb{Z} : [\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{m+n}$$

such that $\dim \mathfrak{g}_n < \infty\ \forall\, n \in \mathbb{Z}$.

(b) $\mathfrak{g}$ *posseses a Lie algebra automorphism ϑ with $\vartheta^2 = \mathrm{id}$,*

$$\vartheta(\mathfrak{g}_n) \subset \mathfrak{g}_{-n}\ \forall\, n \in \mathbb{Z} \quad \textit{and} \quad \vartheta|_{\mathfrak{g}_0} = -\mathrm{id}_{\mathfrak{g}_0}$$

(c) $\mathfrak{g}$ *posseses a symmetric, $\mathfrak{g}$-invariant bilinear form $\langle\, \cdot\,, \cdot\, \rangle$ such that*

$$\forall\, X, Y \in \mathfrak{g} : (X, Y) := -\langle X, \vartheta Y \rangle$$

is a symmetric bilinear form which is positive definite on $\mathfrak{g}_n$ for $n \in \mathbb{Z} \setminus \{0\}$ and for all $n, m \in \mathbb{Z}$ with $m \neq n$

$$\forall\, X \in \mathfrak{g}_m, Y \in \mathfrak{g}_n : (X, Y) = 0$$

Then $\mathfrak{g}$ is a Borcherds-Kac-Moody algebra.

Remark 2.5.8.

(0) Borcherds proves the above theorem by inductively constructing the e_j, f_j, h_{jj} (induction on $|n|$ in $\mathfrak{g}_n$, with $n \neq 0$), using that $\mathfrak{g}_n \perp \mathfrak{g}_m$ if $n \neq m$. Note that if e_j is known, then

$$f_j = -\vartheta e_j \quad \text{and} \quad h_{jj} = [e_j, f_j]$$

(1) For any real Lie algebra $\mathfrak{g}$, conditions (a) - (c) in Theorem 2.5.7 imply:

(i) $\mathfrak{g}_0$ is Abelian. To see this, first note that the grading implies for all $X, Y \in \mathfrak{g}_0$

$$[X, Y] \in \mathfrak{g}_0.$$

Hence, the second condition from (b) yields

$$\vartheta\,[X, Y] = -\,[X, Y]\,.$$

On the other hand, since ϑ is a Lie algebra endomorphism

$$\vartheta\,[X, Y] = [\vartheta X, \vartheta Y] = [-X, -Y] = [X, Y]$$

Thus, $[X, Y] = \vartheta\,[X, Y] = -\,[X, Y]$ and therefore $[X, Y] = 0$.

(ii) $\langle\,\cdot\,,\,\cdot\,\rangle$ is invariant under ϑ :

$$\begin{aligned}\forall\, X, Y \in \mathfrak{g} : \langle \vartheta X, \vartheta Y\rangle &\overset{(c)}{=} -\big(\vartheta X, \vartheta^2 Y\big)\\ &\overset{(b),(c)}{=} -(Y, \vartheta X)\\ &\overset{(c)}{=} \langle Y, X\rangle = \langle X, Y\rangle\end{aligned}$$

(2) For any real Lie algebra $\mathfrak{g}$ with properties (a) - (c) of Theorem 2.5.7, consider the set

$$\mathfrak{r} := \{X \in \mathfrak{g} \mid \langle X, Y\rangle = 0 \;\forall\, Y \in \mathfrak{g}\}$$

called the radical or kernel of $\langle\,\cdot\,,\,\cdot\,\rangle$. Then $\mathfrak{r} \subset \mathfrak{g}_0$, since $\langle\,\cdot\,,\,\cdot\,\rangle$ is non-degenerate outside $\mathfrak{g}_0$, so in particular $\mathfrak{r}$ is an Abelian Lie subalgebra (by (1)). It is in fact central: Let $R \in \mathfrak{r}$, $n \in \mathbb{Z} \setminus \{0\}$, $X \in \mathfrak{g}_n$, $Y \in \mathfrak{g}$, $Z \in \mathfrak{g}_0$. Then by (1)

$$[R, Z] = 0.$$

Moreover, the defining property of $\mathfrak{r}$ entails

$$\langle \underbrace{[R, X]}_{\in \mathfrak{g}_n}, Y\rangle \overset{\mathfrak{g}\text{-inv.}}{=} \langle R, [X, Y]\rangle \overset{R \in \mathfrak{r}}{=} 0.$$

Because Y is arbitrary and $\langle\,\cdot\,,\,\cdot\,\rangle$ is non-degenerate on $\mathfrak{g}_n$, this implies

$$[R, X] = 0.$$

Theorem 2.5.9. (Denominator Identity) [Borcherds88] *Let A denote an irreducible Borcherds-Kac-Moody matrix, $\mathfrak{g} = \mathfrak{g}_{\mathrm{BKM}}(A)$ with notations as in Definition 2.5.3, Theorem 2.5.4. Consider $\varrho \in \mathfrak{h}^*$ such that $\forall\, j \in I$:*

$$\langle \varrho, \alpha_j\rangle = \frac{1}{2}\,\langle \alpha_j, \alpha_j\rangle\,.$$

Then

$$\prod_{\alpha \in \Delta^+} (1 - \mathrm{e}^{-\alpha})^{\dim \mathfrak{g}_\alpha} = \sum_{\sigma \in W} \det(\sigma)\, \mathrm{e}^{\sigma(\varrho) - \varrho} \sum_{n=0}^{\infty} (-1)^n \sum_{\beta \in B_n} \mathrm{e}^{-\sigma(\beta)}$$

$$\textit{with} \quad B_n := \left\{ \alpha_{j_1} + \ldots + \alpha_{j_n} \;\middle|\; \begin{array}{l}\alpha_{j_k} \text{ are imaginary simple roots,}\\ \text{that are pairwise perpendicular}\end{array} \right\}$$

on the domain in $\mathfrak{h}$ where both sides converge.

In particular, $B_0 = \{0\}$ and $B_1 = \{\alpha_j \mid j \in I, a_{jj} \neq 2\}$.

2.6 From $\mathfrak{sl}_2(\mathbb{C})$ to the C in CFT

[Schottenloher2008] is the recommended reference for this section.

Remark 2.6.1. Let $\mathfrak{sl}_2(\mathbb{C}) = \operatorname{span}_{\mathbb{C}}\{e, f, h\}$ where *04 Dec 2017*

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and also consider $\mathfrak{sl}_2(\mathbb{R}) := \operatorname{span}_{\mathbb{R}}\{e, f, h\}$. By Example 2.1.4, $\mathfrak{sl}_2(\mathbb{C})$ is the associated Lie algebra of $\mathrm{SL}_2(\mathbb{C})$ and similarly $\mathfrak{sl}_2(\mathbb{R})$ is associated to $\mathrm{SL}_2(\mathbb{R})$.
To see the latter statement, use the Iwasawa decomposition: $\forall\, A \in \mathrm{SL}_2(\mathbb{R}):$

$$\exists!\,(R, H, E) \in \mathrm{SO}(2) \times \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \,\middle|\, \lambda \in \mathbb{R}_{>0} \right\} \times \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{R} \right\} : A = RHE$$

For $t \in (-\varepsilon, \varepsilon)$,

$$E(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ H(t) = \begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix},\ R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

yield curves in $\mathrm{SL}_2(\mathbb{R})$ with $E(0) = H(0) = R(0) = \mathrm{id}$. Differentiation returns

$$\dot{E}(0) = e,\ \dot{H}(0) = h,\ \dot{R}(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = f - e$$

Thus[32], the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$ is $\mathfrak{sl}_2(\mathbb{R})$ as claimed. Moreover, adding $\tilde{E}(t) = E(\imath t)$, etc., we can generate $\mathfrak{sl}_2(\mathbb{C})$.

Typically, we think of $\mathrm{SL}_2(k)$ $(k \in \{\mathbb{R}, \mathbb{C}\})$ in terms of their representations.

Definition 2.6.2. Let G denote a group.

(1) A representation of G is a group homomorphism

$$\varrho : G \to \mathrm{GL}(V)$$

for some vectorspace V. We say: "G *acts* on V", "V is a G-*module*" or "V is a representation of G". We write for $\gamma \in G$, $v \in V$

$$\gamma.v \quad \text{or} \quad \gamma(v)$$

instead of $\varrho(\gamma)(v)$.

(2) A (left) action of G on a set S is a map

$$\varrho : G \times S \to S$$

such that $\forall\, s \in S : \varrho(\mathrm{id}, s) = s$ and

$$\forall\, \gamma, \tilde{\gamma} \in G : \varrho(\gamma, \varrho(\tilde{\gamma}, s)) = \varrho(\gamma\tilde{\gamma}, s).$$

Shorthand notations are

$$\varrho(\gamma, s) = \gamma.s = \gamma(s)$$

[32] For $a, b, c \in \mathbb{R}$, then consider $\gamma(t) = R(at)H(bt)E(ct)$ and obtain $\dot{\gamma}(0) = a(f - e) + bh + ce$. Hence, $\operatorname{span}_{\mathbb{R}}\{e, f, h\}$ lies in the Lie algebra associated to $\mathrm{SL}_2(\mathbb{R})$. Equality then follows from considering the dimension of $\mathrm{SL}_2(\mathbb{R})$ via the regular value theorem.

(3) Analogously, a RIGHT ACTION is a map

$$S \times G \to S : (s, \gamma) \mapsto s.\gamma$$

with $\forall\, s \in S, \gamma, \tilde{\gamma} \in G$:

$$(s.\tilde{\gamma}).\gamma = s.(\tilde{\gamma}\gamma) \quad \text{and} \quad s.\mathrm{id} = s$$

Example 2.6.3.

(1) For $G = \mathrm{SL}_2(k)$ ($k \in \{\mathbb{R}, \mathbb{C}\}$), consider

(a) the representation of G on k^2 given by $\mathrm{SL}_2(k) \hookrightarrow \mathrm{GL}_2(k)$, the DEFINING REPRESENTATION.

(b) the action of G on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by Möbius transforms (recall Definition 0.2).

(2) For any Lie group G with Lie algebra $\mathfrak{g} = T_{\mathrm{id}}G$ (cf. Remark 2.1.3), we have two left actions $G \times TG \to TG$: on the tangent bundle

$$\begin{aligned} &(\gamma, X) \mapsto \gamma.X \quad\ \ \in T_{\gamma\tilde{\gamma}}G \quad \text{if} \quad X \in T_{\tilde{\gamma}}G \\ \text{and} \quad &(\gamma, X) \mapsto X.\gamma^{-1} \in T_{\tilde{\gamma}\gamma^{-1}}G \quad \text{if} \quad X \in T_{\tilde{\gamma}}G \end{aligned}$$

Thus, we can construct the ADJOINT REPRESENTATION

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) : \gamma \mapsto \mathrm{Ad}_\gamma, \quad \mathrm{Ad}_\gamma(X) := \gamma.X.\gamma^{-1}$$

As an exercise, it can be shown that Ad defines indeed a representation. If you like, restrict to matrix Lie algebras.

Remark 2.6.4. The idea of this remark is that if G acts by "symmetries", then its Lie algebra should act by "infinitesimal symmetries".

(0) If $F : M \to N$ is a smooth map between smooth manifolds, then $\mathrm{d}F : TM \to TN$ obeys:

$$\forall\, X, Y \in \Gamma(M, TM) : \mathrm{d}F([X, Y]) = [\mathrm{d}F(X), \mathrm{d}F(Y)]$$

The proof is left as an exercise.

(1) Let now G denote a Lie group, $\mathfrak{g}$ its Lie algebra and consider a representation $\varrho : G \to \mathrm{GL}(V)$ of G on some vectorspace V with smooth ϱ. Then, by (0), $\mathrm{d}\varrho : \mathfrak{g} \to \mathrm{End}(V)$ is a representation of $\mathfrak{g}$.
As an exercise, consider $\varrho = \mathrm{Ad}$ and show $\mathrm{d}\varrho = \mathrm{ad}$ for any matrix Lie algebra G.

(2) (a) Let G denote a Lie group which acts on a smooth manifold M. Let also $V := C^\infty(M, \mathbb{C})$ and regard

$$\varrho : G \to \mathrm{GL}(V) : \forall\, f \in V, \gamma \in G : \forall\, z \in M : (\varrho(\gamma)(f))(z) := f(\gamma^{-1}.z)$$

Check as an exercise, that ϱ is a representation of G. It is called the INDUCED REPRESENTATION and via (1), $\mathrm{d}\varrho$ gives rise to a representation of $\mathfrak{g}$. Explicitly, for $X \in \mathfrak{g}$, $X = \dot{\gamma}_0$ where $(-\varepsilon, \varepsilon) \to G : t \mapsto \gamma_t$ (with $\gamma_0 = \mathrm{id}$) and $f \in V, z \in M$, we have

$$\begin{aligned} (\mathrm{d}\varrho(X)(f))(z) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma_t^{-1}.z) \\ &= \mathrm{d}f_z(\widehat{X}_z) = \widehat{X}_z(f) \end{aligned}$$

where $\widehat{X}_z = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\gamma_t^{-1}.z) \in T_zM$.

(b) As a simplification of the above, consider $M = \mathbb{R}^m$. Define

$$\forall\, z \in M : \delta\gamma(z) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\gamma_t.z)$$

Then observe $(\mathrm{d}\varrho(X)(f))(z) = -\mathrm{d}f(\delta\gamma(z))$.
For $M = \mathbb{C} = \mathbb{R}^2$, lets generalize to any family $(\gamma_t)_{t\in(-\varepsilon,\varepsilon)}$ of smooth maps $\gamma_t : M \to M$ analytic in t and with $\gamma_0 = \mathrm{id}$. For $t \to 0$, there is an expansion

$$\gamma_t(z) = z + t \cdot \delta\gamma(z) + \mathcal{O}(t^2)$$

Write $z = x + \imath y$ with real coordinates x, y. Then

$$\begin{aligned}
\mathrm{d}f(\gamma_t(z)) &= \frac{\partial f}{\partial x}(z) \cdot \mathrm{Re}\underbrace{\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma_t(z)\right)}_{\delta\gamma(z)} + \frac{\partial f}{\partial y}(z) \cdot \mathrm{Im}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma_t(z)\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath\frac{\partial}{\partial y}\right) f \cdot \delta\gamma(z) + \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y}\right) f \cdot \overline{\delta\gamma(z)} \\
&= \partial_z(f) \cdot \delta\gamma(z) + \partial_{\overline{z}}(f) \cdot \overline{\delta\gamma(z)}
\end{aligned}$$

where $\partial_z, \partial_{\overline{z}}$ are the Wirtinger symbols. So lastly, we get

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma_t(z)) = \tilde{X}_z(f) \quad \text{with} \quad \tilde{X}_z = \delta\gamma(z)\partial_z + \overline{\delta\gamma(z)}\partial_{\overline{z}}$$

meaning $\mathrm{d}\varrho(X) = -\tilde{X}_z$.

(c) Consider the example of $\mathrm{SL}_2(\mathbb{R})$ acting on $\overline{\mathbb{C}}$ by Möbius transforms (Example 2.6.3 (b)). Recalling the notations of Remark 2.6.1, consideration of $\gamma_t(z) = E(t)$ yields

$$\gamma_t(z) = E(t).z = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.z = z + t$$

and thus $\delta\gamma(z) = 1$, $\tilde{X}_z = \partial_z + \partial_{\overline{z}}$. Analogously, we get

$$\begin{aligned}
H(t).z &= \mathrm{e}^{2t} z \\
&\implies \delta\gamma(z) = 2z, \qquad \tilde{X}_z = 2z\partial_z + 2\overline{z}\partial_{\overline{z}} \\
R(t).z &= \frac{\cos(t)z - \sin(t)}{\sin(t)z + \cos(t)} \\
&\implies \delta\gamma(z) = -1 - z^2, \quad \tilde{X}_z = -\partial_z - \partial_{\overline{z}} - z^2\partial_z - \overline{z}^2\partial_{\overline{z}}
\end{aligned}$$

For $\mathrm{SL}_2(\mathbb{C})$, three more vectorfields can be obtained from $\tilde{E}(t) = E(\imath t)$ et cetera. Note that defining

$$l_n = -z^{1-n}\partial_z, \quad \bar{l}_n = -\overline{z}^{1-n}\partial_{\overline{z}}$$

for $n \in \{0, \pm 1\}$ yields

$$\begin{aligned}
\mathrm{d}\varrho(e) &= l_1 + \bar{l}_1, & \mathrm{d}\varrho(h) &= 2l_0 + 2\bar{l}_0, \\
\mathrm{d}\varrho(f) &= -l_{-1} + \bar{l}_{-1}, & \mathrm{d}\varrho(\imath e) &= \imath l_1 - \imath\bar{l}_1, \quad \text{etc.}
\end{aligned}$$

Recall the exercise after Remark 2.4.5, where we had an action of the Witt algebra on $\mathbb{C}\left[z, z^{-1}\right]$ via $l_n = -z^{1-n}\partial_z$ for $n \in \mathbb{Z}$.
The above reveals the action of $l_0, l_{\pm 1}$ as yielding infinitesimal symmetries.

The next task is to understand the action by Möbius transforms in terms of symmetries and to extend from $\mathfrak{sl}_2(\mathbb{C})$ to all of the Witt algebra.

06 Dec 2017 **Definition 2.6.5.** Consider two Riemannian manifolds (M, g), $(\tilde{M}, \tilde{g})$. A smooth map $\gamma : M \to \tilde{M}$ is CONFORMAL iff $\forall\, p \in M : \forall\, v, w \in T_pM$:

$$\tilde{g}_{\gamma(p)}(\mathrm{d}\gamma_p(v), \mathrm{d}\gamma_p(w)) = g_p(v, w)\,(\lambda(p))^2$$

for some smooth map $\lambda : M \to \mathbb{R}_{>0}$, i.e. $\gamma^*\tilde{g} = \lambda^2 g$. If $M \subset \tilde{M}$ and $\gamma(M) \cap M$ is dense in $\tilde{M}$, then γ is a CONFORMAL SYMMETRY. If γ is a conformal symmetry and admits a continuous injective extension to $\overline{M} = M \cup S$, $S \subset \overline{M}$ finite, then γ is a GLOBAL CONFORMAL SYMMETRY.

Example 2.6.6.

(1) Let $\tilde{M} = \mathbb{R}^n$ equipped with the Euklidean standard metric g and set $\|x\|^2 = g(x, x)$ for $x \in \tilde{M}$. Consider the following types of maps

  (a) ISOMETRIES: $M = \mathbb{R}^n = \overline{M}$, $\forall\, A \in \mathrm{O}(n), b \in \mathbb{R}^n$:

$$\forall\, x \in M : \gamma(x) = Ax + b$$

  (b) DILATIONS: $M = \mathbb{R}^n = \overline{M}$, $\forall\, \lambda \in \mathbb{R}_{>0}$:

$$\forall\, x \in M : \gamma(x) = \lambda x$$

  (c) SPECIAL CONFORMAL TRANSFORMATIONS $\forall\, c \in \mathbb{R}^n \setminus \{0\}$, $M = \mathbb{R}^n \setminus \left\{-\frac{c}{\|c\|^2}\right\}$, $\overline{M} = \mathbb{R}^n \cup \{\infty\}$

$$\forall\, x \in M : \gamma(x) = \frac{x + \|x\|^2 c}{1 + 2g(x, c) + \|x\|^2 \|c\|^2}$$

  As an exercise, show that all the above are global conformal symmetries.

(2) Consider $\tilde{M} = \mathbb{C} \cong \mathbb{R}^2$ with standard Euklidean metric: $\forall\, z_1, z_2 \in \mathbb{C}$, $z_j = x_j + \imath y_j, j \in \{1, 2\}$:

$$g(z_1, z_2) = x_1x_2 + y_1y_2 = \frac{1}{2}(\overline{z_1}z_2 + z_1\overline{z_2})$$

  (A) (1)(a) becomes $\gamma(z) = az + b = \begin{pmatrix} a^{1/2} & a^{-1}b \\ 0 & a^{-1/2} \end{pmatrix}.z$ for $a, b \in \mathbb{C}$ with $|a| = 1$.

  (1)(b) becomes $\gamma(z) = \lambda z = \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}.z$ for $\lambda \in \mathbb{R}_{>0}$.

  (1)(c) becomes $\gamma(z) = \frac{z}{\overline{c}z+1} = \begin{pmatrix} 1 & 0 \\ \overline{c} & 1 \end{pmatrix}.z$ for $c \in \mathbb{C} \setminus \{0\}$.

  All of these are global conformal symmetries.

  (B) On $M = \mathbb{C}^*$, $\gamma(z) = z^k \forall\, k \in \mathbb{Z} \setminus \{0\}$ is conformal since $\mathrm{d}\gamma_z(v) = kz^{k-1}v$ and thus

$$g_\gamma(z)(\mathrm{d}\gamma_z(v), \mathrm{d}\gamma_z(w)) = \underbrace{\left|kz^{k-1}\right|^2}_{(\lambda(z))^2} g_z(v, w)$$

  However, γ is a global conformal symmetry if and only if $|k| = 1$. The same holds for $\overline{\gamma}(z) = \overline{z}^k$, $k \in \mathbb{Z} \setminus \{0\}$.

Note that the maps in (A) generate the action of $\mathrm{SL}_2(\mathbb{C})$ on $\overline{\mathbb{C}}$, while those in (B) do not form a group.

In the following, we always equip $\mathbb{R}^n$ and $\mathbb{C}$ with the standard Euklidean metric.

Theorem 2.6.7.

(1) *If $n > 2$, then every conformal map $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ is a global conformal symmetry. The group of conformal symmetries of $\mathbb{R}^n$ is generated by the isometries, dilations and special conformal transformations.*

(2) *For $M = \mathbb{C} = \mathbb{R}^2$*

  (a) [Gauss 1825] *If $U \subset M$ is open, then for $\gamma : U \to \mathbb{C}$ differentiable, γ is conformal iff either γ is holomorphic (i.e. $\partial_{\overline{z}}\gamma = 0$) and $\partial_z\gamma \neq 0$ everywhere, or $\overline{\gamma}$ obeys these conditions.*

  (b) *The orientation preserving global conformal symmetries of $\overline{\mathbb{C}}$ are precisely the Möbius transforms.*

Definition 2.6.8. For open $U \subset \mathbb{C}^*$, and a family $(\gamma_t)_{t\in(-\varepsilon,\varepsilon)}$ of conformal maps analytic in t and with $\gamma_0 = \mathrm{id}$, the ASSOCIATED INFINITESIMAL CONFORMAL TRANSFORMATION (of U) is

$$\begin{aligned} X_z &= \frac{\mathrm{d}}{\mathrm{d}t}\gamma_t(z)\Big|_{t=0}\partial_z + \overline{\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t(z)\Big|_{t=0}}\partial_{\overline{z}} \\ &= \delta\gamma(x)\partial_z + \overline{\delta\gamma(z)}\partial_{\overline{z}} \end{aligned}$$

Note. The infinitesimal conformal transformations generate a complex Lie algebra with "topological" basis

$$\left(l_n = -z^{1-n}\partial_z,\ \bar{l}_n = -z^{1-n}\partial_{\overline{z}},\ n \in \mathbb{Z}\right)$$

Only the "real" linear combinations of expressions of the types

$$l_n + \bar{l}_n \quad \text{and} \quad \imath l_n - \imath \bar{l}_n$$

yield infinitesimal conformal transformations (associated to $\gamma_t(z) = z - tz^{1-n}$, $\tilde{\gamma}_t(z) = z - \imath tz^{1-n}$). Consider the generating Lie brackets: $\forall\, n, m \in \mathbb{Z}$:

$$[l_n, l_m] = (m-n)l_{m+n}, \quad \left[\bar{l}_n, \bar{l}_m\right] = (m-n)l_{m+n}, \quad \left[l_n, \bar{l}_m\right] = 0$$

- We always work with the WITT ALGEBRA, the complex Lie algebra with (Hamel) basis $(l_n)_{n\in\mathbb{Z}}$ over $\mathbb{C}$ and its "rightmoving" partner with basis $(\bar{l}_n)_{n\in\mathbb{Z}}$.
- In contrast, many authors use either[33]

  $$\mathrm{Der}(\mathbb{C}(\!(z)\!)) = \mathbb{C}(\!(z)\!)\partial_z$$

  or

  $$\mathbb{C} \otimes \mathrm{Vect}(S^1) = \mathbb{C} \otimes \mathfrak{X}(S^1)$$

  and examples are obtained by restricting the l_n to S^1.

In CFT, the space of states is a representation of a central extension of the Witt algebra and its rightmoving partner.

[33] $\mathbb{C}(\!(z)\!) := \left\{\sum_{n=-N}^{\infty} a_n z^n \,\middle|\, N \in \mathbb{Z}, a_n \in \mathbb{C}\right\}$ is the space of formal Laurent series with coefficients in $\mathbb{C}$.

Definition 2.6.9.

(1) The WITT ALGEBRA W is the complex Lie algebra with basis $(l_n)_{n\in\mathbb{Z}}$ and the Lie bracket defined by

$$\forall\, n,m\in\mathbb{Z} : [l_n, l_m] = (m-n)l_{m+n}$$

(2) The VIRASORO ALGEBRA is the complex Lie algebra with basis $((L_n)_{n\in\mathbb{Z}}, \mathbf{1})$ with the Lie bracket such that $\mathbf{1}$ is central and $\forall\, n,m\in\mathbb{Z}$:

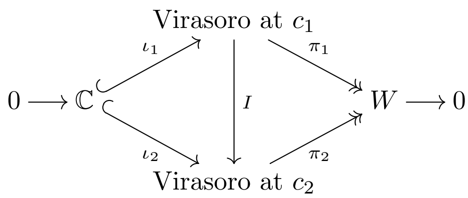
$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\mathbf{1}$$

for some $c\in\mathbb{C}$, the CENTRAL CHARGE.

Remark 2.6.10.

(1) The Virasoro algebra at central charge c is a central extension of the Witt algebra by $\mathbb{C}$ (exercise).

(2) As Lie algebras over $\mathbb{C}$, all Virasoro algebras at central charge $c\neq 0$ are isomorphic. However, at central charges $c_1\neq c_2$, the two corresponding Virasoro algebras are not equivalent as extensions of W: There is no Lie algebra homomorphism I such that the following diagram commutes:



Show this as an exercise.

(3) Every central extension of W by $\mathbb{C}$ is equivalent to some Virasoro algebra at central charge c.

(4) With notations as in Definition 2.6.9,

$$\forall\, m,n\in\{0,\pm1\} : [L_n, L_m] = (m-n)L_{m+n}$$

holds. So if we introduce a rightmoving partner with basis $((\overline{L}_n)_{n\in\mathbb{Z}}, \overline{\mathbf{1}})$, Remark 2.6.4 and Example 2.6.6 yield

$$\begin{array}{rcl}
\operatorname{span}_{\mathbb{R}}\left\{L_1+\overline{L}_1, \imath L_1 - \imath\overline{L}_1\right\} & \leftrightarrow & \text{translations}\\
\operatorname{span}_{\mathbb{R}}\left\{L_0, \overline{L}_0\right\} & \leftrightarrow & \text{dilations}\\
\operatorname{span}_{\mathbb{R}}\left\{\imath L_0, \imath\overline{L}_0\right\} & \leftrightarrow & \text{rotations}\\
\operatorname{span}_{\mathbb{R}}\left\{L_{-1}+\overline{L}_{-1}, \imath L_{-1} - \imath\overline{L}_{-1}\right\} & \leftrightarrow & \text{special conformal transformations}
\end{array}$$

Exercise 2.6.11. Consider the complex Lie algebra $\mathfrak{A}$ with basis $((a_n)_{n\in\mathbb{Z}}, \mathbf{1})$ and the Lie bracket such that $\mathbf{1}$ is central and $\forall\, m,n\in\mathbb{Z}$:

$$[a_n, a_m] = m\delta_{m+n,0}\mathbf{1}.$$

It is called the HEISENBERG ALGEBRA[34].
Assume that we have a representation of $\mathfrak{A}$ on some vectorspace $\mathbb{H}$ (such that we have $a_n \in \mathrm{End}_{\mathbb{C}}(\mathbb{H})$ and $\mathbf{1}$ is represented by id). Then, ignoring any issues of convergence, consider

$$L_n := \frac{1}{2} \sum_{k=-\infty}^{\infty} a_{n+k} a_{-n}.$$

For $m, n \in \mathbb{Z} \setminus \{0\}$ with $m + n \neq 0$, show

$$[L_n, L_m] = (m-n) L_{m+n}$$

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[34] It is associated to the symplectic vectorspace with basis $(a_n)_{n\in\mathbb{Z}}$ and symplectic form $\omega(a_n, a_m) = m\delta_{m+n,0}$. For the rest of the lecture, this is *the* Heisenberg algebra.

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