GRK 1821 WEDNESDAY SEMINAR, WINTER TERM 2015/16

K3 SURFACES

1. Generalities; good educational background

1.1 (I). Introduction to complex geometry (*21.10.2015*).

[Jens Eberhardt]

Introduce almost complex structures on smooth manifolds M, the (p, q)-forms $\mathcal{A}^{p,q}(M)$, and the operators d, ∂, $\overline{\partial}$. State (and maybe prove) the $\overline{\partial}$-Poincaré lemma. Explain the integrability condition for almost complex structures, including the Newlander-Nirenberg Theorem, to make contact with the definition of complex manifolds by holomorphic atlases. Introduce holomorphic vector bundles (don't forget to discuss short exact sequences of such bundles!), in particular the holomorphic tangent bundles of complex manifolds, and Dolbeault cohomology.

For guidance, [Wen15, §1.1, up to the first paragraph of page 6] gives a brief summary. A useful reference is [Huy05, §2.6], including some of the material leading up to this section. For the discussion of integrability, [Wei71, II 2&3] is excellent and should be used, see in particular [Wei71, II Prop. 2]. For Dolbeault cohomology, use [BHPvdV04, I §12]; include their Lemma 12.1 (without proof) or agree with the speaker for talk (III) that they will do so.

1.2 (II). Introduction to Kähler geometry (*28.10.2015*).

[Konrad Voelkel]

Define Kähler manifolds, discussing various equivalent definitions. At least, you should include the following characterizations: (i) closedness of the Kähler form; (ii) existence of local Kähler potentials; (iii) agreement of Levi-Civita and Chern connection; (iv) almost complex structure, which is parallel with respect to the Levi-Civita connection; (v) osculation of the metric.

Introduce the Hodge star and the Lefschetz operators, and discuss the Kähler identities (don't attempt to give a full proof!).

For guidance, [Wen15, §1.1, p.6–Lemma 1.1.2] gives a brief summary. Useful references are [Bal, §4] and [Huy05, §3.1]. You may also find [Wei71, I&II] helpful.

1.3 (III). Hodge Theory and Chern-Weil Theory on Kähler manifolds (*04.11.2015*).

[Sebastian Goette]

In the first half of the talk, discuss the behaviour of the Fröhlicher spectral sequence for Kähler manifolds, the Hodge decomposition, and the $\partial\overline{\partial}$-lemma. Useful references include [Huy05, §3.2] and [BHPvdV04, I (13.2)–(13.7)]. You may also find [Wei71, I&II] helpful.

In the second half of the talk, give an introduction to Chern-Weil Theory, to pave the way for the index theory approach of talk (VI). References on this topic include [MS74, Appendix C] and [Hir66, Chapter I].

1.4 (IV). **A compact complex surface is Kähler if and only if its first Betti number is even (*11.11.2015*).** [Oliver Bräunling and Behrouz Taji]

Summarize the proof of this statement by introducing and constructing Kähler currents.

The proof is due to Lamari [Lam99], and it uses Demailly's regularity theorem [Dem92], in order to show that the existence of Kähler currents guarantees the Kähler condition for some (repeated) blow up of the surface. A criterion for the existence of Kähler currents is then derived in terms of so-called weakly plurisubharmonic $(1, 1)$ currents. If the first Betti number is even, then one proves a strengthening of the $\partial\overline{\partial}$ lemma, which finally leads to the completion of the proof.

An extended summary with simplifications from [DP04] is given in [BHPvdV04, IV §3]. The proof is ingeneous, and certainly a little challenging.

1.5 (V). **The Calabi-Yau Theorem (*18.11.2015*).** [Anda Degeratu]

The Calabi-conjecture [Cal54], proved by S.-T. Yau in [Yau78], states the following: Let M denote a Calabi-Yau manifold, i.e. a compact Kähler manifold with $c_1(M) = 0$. Then every Kähler class of M contains a unique Ricci-flat Kähler metric.

Introduce Calabi-Yau manifolds and discuss the condition of vanishing first Chern class and how it relates to the Ricci tensor. Explain the statement of the theorem by introducing Kähler classes. Give an outline of the proof by stating the complex Monge-Ampère equation, and by summarizing Yau's proof of existence of solutions.

2. K3 surfaces: Definition, topological invariants, and examples

2.1 (VI). **K3 surfaces: Definition and classical invariants (*25.11.2015*).** [Maximilian Schmidtke and Anja Wittmann]

Define K3 surfaces and calculate their classical topological and analytic invariants, namely the Euler number, signature, and holomorphic Euler characteristic ($\hat{A}$-genus), and then the cohomology over $\mathbb{R}$. See [BHPvdV04, VIII §2] for the definition and [BHPvdV04, VIII (3.1)–(3.5)] for the results, but preferably follow [Wen15, §1.2] and references therein to include the classification of Calabi-Yau 2-folds via the Atiyah-Singer Index theorem in your talk: Every connected Calabi-Yau 2-fold is either a complex torus or a K3 surface. Here, [Hir66, Chapters II and III] may be helpful. Do not discuss the integral cohomology of K3 [BHPvdV04, VIII (3.3)], since this is done in talk (VII).

2.2 (VII). **The integral cohomology of K3 (*02.12.2015*).** [Nadine Grosse]
Present the calculation of the integral cohomology of K3, following, for example, [BHPvdV04, VIII (3.3)]. You will need to intoduce some background material from lattice theory, which you can find in [BHPvdV04, I §2] or in [Mor84, §1].

Introduce the notions of Néron-Severi lattice, transcendental lattice, and Picard number ([BHPvdV04, VIII §1] and [Mor84, §1]). Using Kodaira's Embedding Theorem [GH78, p. 191], show that a K3 surface is algebraic (or, equivalently, projective) if and only if its Néron-Severi lattice is not negative definite; see also the notion of polarized Hodge structure in [Mor84, §1]. Introduce the notions of effective divisors, the Kähler cone, effective Hodge isometries and Picard-Lefschetz reflections, using [BHPvdV04, VIII §1] and covering [BHPvdV04, VIII (3.6)–(3.12)]. Using Kodaira's embedding theorem again (see also [BHPvdV04, IV (6.1)]), show that all rational elements in the Kähler cone are Kähler classes.

If time allows, it would be nice to characterize minimally resolved surface singularities on a K3 surface through the occurence of nodal classes in the Néron-Severi lattice.

While the first few pages of [Mor84] are very helpful for this talk, note that the surjectivity of the period map, [Mor84, Theorem 1.6], is extensively used in that paper, but that we will prove it only much later in the seminar.

2.3 (VIII). **The Kummer construction (*09.12.2015*).** [Florian Beck]
Introduce the Kummer construction and show that every Kummer surface is a K3 surface, following [BHPvdV04, V §16], [Wen15, §1.3]. You may find [BHPvdV04, I §9] useful. Present the calculation of the Néron-Severi lattice for Kummer surfaces [BHPvdV04, VIII (5.1)-(5.8)], including the relevant background material on affine geometry over $\mathbb{F}_2$ from [BHPvdV04, VIII §4]. Include the characterization of Kummer K3s through the occurence of 16 disjoint nodal curves [BHPvdV04, VIII (6.1)], stated also at the end of [Wen15, §1.3], or agree with the next speaker that they will do so.

You may find parts of [Mor84, §2–4] useful, though this reference occasionally uses the surjectivity of the period map, which we will prove only much later in the seminar.

If time allows, you can include further material on surface singularities that can occur on K3 (coordinate this with the previous speaker!), or you can explain how, by a construction which is analogous to the Kummer construction, one obtains manifolds with G_2 holonomy [Joy00, §§11, 12].

2.4 (IX). **The Torelli Theorem for Kummer K3s (*16.12.2015*).**
[Martin Schwald]
The Torelli Theorem for Kummer surfaces is a main ingredient that we will need in order to construct the moduli space of complex structures on K3. It is beautiful in its own right, as we shall see in this talk:

State and prove the Torelli Theorem and the weak Torelli Theorem for projective Kummer surfaces, following [BHPvdV04, VIII (5.9)] and [BHPvdV04, VIII §6]. You should agree with the previous speaker whether or not they are already covering the characterization of Kummer K3s [BHPvdV04, VIII (6.1)]. In the proof of [BHPvdV04, VIII (5.9)], you will need a version of the Torelli theorem for complex two-tori, [BHPvdV04, I (14.2) and V §3], which you should also present. Very nice related results can be found in [SM74], and if time allows, you may want to mention them.

2.5 (X). **K3 generates the 4-dimensional spin cobordism group (*23.12.2015*).**
[Yi-Sheng Wang]

Introduce the spin cobordism group Ω_4^{spin} and explain why it is isomorphic to $\mathbb{Z}$ with K3 as generator. This is the statement of Corollary 1 in [Kir89, XI §1]. The main task for the speaker will be to give a lightning (and enlightening) introduction to cobordism theory, to motivate why this result is useful, and to summarize its proof.

3. Moduli spaces of K3 surfaces

3.1 (XI). All K3 surfaces are diffeomorphic (*13.01.2016*).

[Natalie Peternell]

The first result to be presented in this talk is the local Torelli theorem [BHPvdV04, VIII (7.3)], which describes the moduli space of complex structures on K3 locally. In particular, the period map is shown to yield a local isomorphism. The proof uses standard techniques from deformation theory; you can follow the arguments in [BHPvdV04, VIII §7].

Secondly, following [BHPvdV04, VIII §8], show that period points of marked projective Kummer surfaces are dense in the period domain [BHPvdV04, VIII (8.5)]. As a corollary, it follows that any two K3 surfaces are diffeomorphic [BHPvdV04, VIII (8.6)], because this is so for all projective marked Kummer surfaces.

3.2 (XII). The moduli space of complex structures on K3 (*20.01.2016*).

[Daniel Harrer and Doris Hein]

This talk and the next one are closely related; the next speaker and you may decide to distribute the material differently from what is suggested here. The aim is to give a global construction of the moduli space of complex structures on K3 by means of the period map. In fact, a refined period map is needed, which is shown to be injective. Its surjectivity will be proved in yet one further step in talk (XIII).

Give the construction of the moduli space of complex structures on K3 surfaces by gluing Kuranishi families as in [BHPvdV04, VIII (12.1)]. Show that this moduli space is non-Hausdorff by Atiyah's example [Ati58], see [BHPvdV04, VIII (12.2)].

Introduce the notions of marked pairs, consisting of a K3 surface and a Kähler class on it, and the refined period map as in [BHPvdV04, VIII §12]. Give the construction of the fine moduli space of marked pairs. This uses results on the behaviour of the Kähler cone under deformations [BHPvdV04, VIII §9], which you should also present.

Conclude by stating the description [BHPvdV04, VIII (12.3)] of the moduli space of complex structures on K3, which will follow from the Torelli theorem, whose proof is the subject of the next talk.

3.3 (XIII). The Torelli Theorem for K3 surfaces (*27.01.2016*).

[Emanuel Scheidegger]

This talk and the previous one are closely related; the previous speaker and you may decide to distribute the material differently from what is suggested here.

State and prove the Torelli theorem for K3 surfaces [BHPvdV04, VIII (11.1)–(11.4)]. The proof uses the results of [BHPvdV04, VIII §10] on degenerations of isomorphisms between K3 surfaces, which you should present, as well.

3.4 (XIV). Hyperkähler geometry and surjectivity of the period map for K3 (*03.02.2016*).

[Katrin Wendland]

This talk completes the description of the moduli space of complex structures on K3, by showing that the refined period map is also surjective. In fact, instead of the moduli space of complex structures, in effect one constructs the moduli space of hyperkähler structures on K3 and then obtains the complex structure moduli space from the latter.

Introduce the notion of hyperkähler structure and discuss its properties, following [BHPvdV04, VIII §13] (where hyperkähler structures are called quaternionic structures). If time allows, you can discuss special holonomies in this context [Huy05, 4.A.12-4.A.18], [Joy00]. Summarize the proof of the surjectivity of the period map for K3 surfaces, following [BHPvdV04, VIII §14]. As an application of the surjectivity of the period map, deduce global descriptions of the moduli spaces of complex structures and hyperkähler structures on K3, respectively, in terms of Grassmannians of positive definite oriented 2-dimensional and 3-dimensional subspaces, respectively, in $H^2(K3, \mathbb{R})$ (see also [BHPvdV04, VIII (8.5) and (13.5)]).

If time allows, discuss lattice polarized K3 surfaces and their moduli spaces [Dol96].

4. K3 surfaces from an arithmetic view point

4.1 (XV). **The Kuga-Satake construction (*10.02.2016*).** [Fritz Hörmann] According to [Huy, §4], the Kuga-Satake construction "associates with any weight two Hodge structure a Hodge structure of weight one. Geometrically, this allows to pass from the K3 surface to a complex torus."

The task of this talk is to give an lightning (and enlightening) overview on the Kuga-Satake construction, starting out over $\mathbb{R}$, see [Huy, §4]. It would be interesting if you explained some arithmetic properties of K3 surfaces and their relation to the Tate conjecture.

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