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Exercise sheet 9

Exercise 35. *More quotient singularities.*

(a) Show that the following affine algebraic varieties

$$\begin{aligned} &V(x_0^2 + x_1^{n-1} + x_1 x_2^2), \quad \text{for } n \geq 3 \\ &V(x_0^2 + x_1^3 + x_2^4) \\ &V(x_0^2 + x_1^3 + x_1 x_2^3) \\ &V(x_0^2 + x_1^3 + x_2^5) \end{aligned}$$

in $\mathbb{A}^3_{\mathbb{C}}$ have a unique singular point at $x = 0$.

(b) The varieties in (a) are "quotient singularities", i.e. they are isomorphic to affine algebraic varieties $(\mathbb{C}^2/G, \mathbb{C}[z_1, z_2]^G)$ for appropriate groups $G \subset \mathrm{SU}(2)$. Show this for the first family of varieties in (a).

Exercise 36. *Complex projective spaces.* Show that for any $n \in \mathbb{N}$, the n-dimensional complex projective space $\mathbb{CP}^n$ is a smooth manifold by constructing an atlas such that each coordinate patch is homeomorphic to an open subset of $\mathbb{C}^n$ and such that all coordinate changes are holomorphic (in each component). This shows that $\mathbb{CP}^n$ is a *complex* manifold.

Exercise 37. *Ideals.*

(a) Prove that all maximal ideals in $\mathbb{C}[z]$ are of the form $(z - a)$ for $a \in \mathbb{C}$.

Hint. You may use the Euclidean algorithm for elements of $\mathbb{C}[z]$.

Note. This result generalizes as follows. All maximal ideals in $\mathbb{C}[z_1, \ldots, z_n]$ are of the form $(z_1 - a_1, \ldots, z_n - a_n)$ for $a_1, \ldots, a_n \in \mathbb{C}$.

(b) Give an example of a non-zero, proper prime ideal $\mathfrak{p}$ of $\mathbb{C}[z_1, z_2]$ which is not maximal and determine all the maximal ideals containing $\mathfrak{p}$.

Exercise 38. *Tangent bundles of complex manifolds.* Let M be a complex manifold M with a holomorphic atlas $\{U_\alpha\}_\alpha$, i.e. M can be covered by open charts $U_\alpha \subset M$ such that for each U_α, there exists a homeomorphism $\phi_\alpha \colon U_\alpha \to V_\alpha$, where $V_\alpha \subset \mathbb{C}^m$ is open, and the coordinate changes

$$\phi_\alpha \circ \phi_\beta^{-1}|_{\phi_\beta(U_\alpha \cap U_\beta)} \colon \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic.

Show the following:

(a) The transition functions for the holomorphic tangent bundle $\mathcal{T}_M$ and the holomorphic cotangent bundle $\mathcal{T}_M^*$ are given on each intersection $U_\alpha \cap U_\beta$ of open charts by holomorphic maps

$$A, B \colon U_\alpha \cap U_\beta \to \mathrm{GL}_m(\mathbb{C})$$

with $AB^T = \mathrm{id}_{\mathbb{C}^m}$.

(b) There is an isomorphism of complex vector bundles $\mathrm{T}M \xrightarrow{\simeq} \mathcal{T}_M$ induced by

$$\begin{aligned} \mathrm{T}M|_U &\xrightarrow{\simeq} \mathcal{T}_M|_U \\ \frac{\partial}{\partial x^j} &\longmapsto \frac{\partial}{\partial z^j} \\ \frac{\partial}{\partial y^j} &\longmapsto i\frac{\partial}{\partial z^j} \end{aligned}$$

where U is a chart with coordinates $z^k = x^k + iy^k$ for $k \in \{1, \dots, m\}$.