Differentialgeometrie II
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https://home.mathematik.uni-freiburg.de/mathphys/lehre/SoSe21/DiffGeoII.html

Exercise sheet 8

Exercise 31. *Generators for simple Lie algebras.* Let $\mathfrak{g}$ be a simple Lie algebra with Cartan subalgebra $\mathfrak{h}$, root system $\Delta = \Delta^+ \cup \Delta^-$ and Π a basis of simple roots. Show that the set $\{E_{\pm\alpha} \mid \alpha \in \Pi\}$ generates $\mathfrak{g}$ as a Lie algebra.

Exercise 32. *Serre relations.* Let $\mathfrak{g}$ be a simple Lie algebra with Cartan subalgebra $\mathfrak{h}$, root system $\Delta = \Delta^+ \cup \Delta^-$ and let $\Pi = (\alpha_1, \ldots, \alpha_r)$ be a basis of simple roots with associated Cartan matrix $A = (a_{ij})_{i,j\in\{1,\ldots,r\}}$.

Show that the relations

$$\begin{aligned} [h_i, h_j] &= 0 & [h_i, x_j] &= a_{ij} x_j \\ [x_i, y_j] &= \delta_{ij} h_j & [h_i, y_j] &= -a_{ij} x_j \end{aligned} \qquad \text{for all } i, j$$

and

$$\begin{aligned} (\mathrm{ad}_{x_i})^{-a_{ij}+1}(x_j) &= 0 \\ (\mathrm{ad}_{y_i})^{-a_{ij}+1}(y_j) &= 0 \end{aligned} \qquad \text{for all } i \neq j$$

hold for an appropriate choice of basis $(h_1, \ldots, h_r)$ of $\mathfrak{h}$ and for suitable $x_j \in \mathfrak{g}_{\alpha_j} \setminus \{0\}$ and $y_j \in \mathfrak{g}_{-\alpha_j} \setminus \{0\}$. Prove that for your choice of x_j and y_j

$$\begin{aligned} (\mathrm{ad}_{x_i})^{-a_{ij}}(x_j) &\neq 0 \\ (\mathrm{ad}_{y_i})^{-a_{ij}}(y_j) &\neq 0. \end{aligned}$$

Exercise 33. *Ideals.* Show the following:

(a) The vanishing ideal $I(Z)$ of an algebraic set $Z \subset \mathbb{A}^n_{\mathbb{C}}$ is an ideal of $\mathbb{C}[z_1, \ldots, z_n]$.

(b) If $Y \subset Z \subset \mathbb{A}^n_{\mathbb{C}}$ are algebraic sets, then $I(Z) \subset I(Y)$.

(c) For an algebraic set $Z \subset \mathbb{A}^n_{\mathbb{C}}$ one has $V(I(Z)) = Z$, where for any ideal $J \subset \mathbb{C}[z_1, \ldots, z_n]$, one defines $V(J) = \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ for all } f \in J\}$. Is it true that $I(V(J)) = J$ for any ideal J?

Exercise 34. *Coordinate change.* Let $n \geq 1$ and let $f \in \mathbb{C}[z_1, \ldots, z_n] \setminus \{0\}$ be any polynomial which is not identically zero. Show that

$$\begin{aligned} &(V(f + x^2 + y^2), \mathbb{C}[x, y, z_1, \ldots, z_n]/(f + x^2 + y^2)) \\ \text{and} \quad &(V(f + xy), \mathbb{C}[x, y, z_1, \ldots, z_n]/(f + xy)) \end{aligned}$$

are isomorphic affine varieties.