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https://home.mathematik.uni-freiburg.de/mathphys/lehre/SoSe21/DiffGeoII.html

Exercise sheet 7

Exercise 26. *Root systems and root reflections.* Let $(V, \langle -,- \rangle)$ be a finite-dimensional Euclidean vector space and let $\Delta \subset V \setminus \{0\}$ be a root system in V (see Definition 1.4.9).

For a root $\alpha \in \Delta$ let

$$\begin{aligned} s_\alpha \colon V &\to V \\ v &\mapsto v - \frac{2\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \end{aligned}$$

denote the corresponding root reflection. Show the following:

(a) For all $\alpha \in \Delta$ we have that

- $s_\alpha \in \mathrm{O}(V, \langle -,- \rangle)$
- $s_\alpha^2 = \mathrm{id}$
- $s_\alpha(\alpha) = -\alpha$
- $s_\alpha(v) = v$ if $\langle \alpha, v \rangle = 0$

so that in particular $-\alpha \in \Delta$.

(b) $\langle s_\alpha \mid \alpha \in \Delta \rangle$ is a finite subgroup of $\mathrm{O}(V, \langle -,- \rangle)$.

Exercise 27. *Isomorphisms of root systems.* Show that an isomorphism of root systems in $(V, \langle -,- \rangle)$ need not be given by an orthogonal transformation of $(V, \langle -,- \rangle)$.

Exercise 28. *Maximal positive roots.* Let $\Delta = \Delta^+ \cup \Delta^-$ be a partition of a root system into positive and negative roots as in Definition 1.4.14. Show that if Δ is irreducible, then there exists a unique maximal $\rho \in \Delta^+$, i.e. for all $\alpha \in \Delta^+$, $\rho + \alpha \notin \Delta^+$.

Exercise 29. Let Δ denote a root system in a Euclidean vector space $(V, \langle -,- \rangle)$ and $\Delta = \Delta^+ \cup \Delta^-$ be a partition into positive and negative roots. Let $\Pi = (\alpha_1, \dots, \alpha_r)$, where $\alpha_j \in \Delta^+$ for $j \in \{1, \dots, r\}$ are simple roots, be a basis of V and let

$$A = (a_{ij})_{1 \le i,j \le r} \quad \text{with} \quad a_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}$$

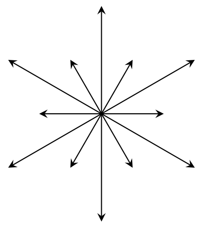
be the corresponding Cartan matrix. Show the following:

(a) For all $i, j \in \{1, \dots, r\}$, $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$ and $a_{ij} \le 0$ if $i \ne j$.

(b) For all $i, j \in \{1, \dots, r\}$, $a_{ij} = 0 \iff a_{ji} = 0$ (note that in general $a_{ij} \neq a_{ji}$).

(c) there exists a matrix $D = \mathrm{diag}(d_1, \dots, d_r)$ with $d_j > 0$ for $j \in \{1, \dots, r\}$ such that DAD^{-1} is symmetric positive definite; if Δ is irreducible, then D is unique up to multiplication by a scalar.

Exercise 30. *The root system of type G_2.* Let Δ be the set of the following 12 vectors in $\mathbb{R}^2$ with the usual Euclidean inner product



where each long arrow is the sum of the neighbouring two shorter arrows.

(a) Show that Δ is an abstract root system and determine whether Δ is reducible or irreducible.

(b) Choose a decomposition $\Delta = \Delta^+ \dot\cup \Delta^-$ into positive and negative roots and choose a basis Π of simple roots. For this choice, calculate the corresponding Cartan matrix (see Exercise 29) and determine the maximal positive root (see Exercise 28).

(c) Determine all possible lengths of the root strings for Δ.