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Exercise sheet 6

Exercise 22. For $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ and $\mathfrak{h}$, α_{ij} as in Exercise 18 and H_α, $\alpha \in \mathfrak{h}^*$, as in Exercise 21, calculate $H_{\alpha_{ij}}$ for any $i, j \in \{0, \ldots, n\}$ with $i \neq j$.

Exercise 23. Give a proof of the following statement, where the Assumptions 1.4.1 are required:

> **Corollary 1.4.7.** Let $\alpha, \beta \in \Delta \cup \{0\}$.
>
> (a) If $\alpha + \beta \neq 0$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
>
> (b) If $\alpha, \beta \neq 0$ and $0 \notin \Delta_\beta^\alpha$, then
>
> $$[E_{-\alpha}, [E_\alpha, E_\beta]] = \frac{n_+}{2}(1 + n_-)\alpha(H_\alpha)E_\beta$$
>
> where $n_\pm$ are as in the proof of Proposition 1.4.6.

Exercise 24. *Low-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.* Let

$$\mathfrak{sl}_2(\mathbb{C}) = \{X \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid \operatorname{tr} X = 0\}$$

with basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall from Theorem 1.4.5 that an n-dimensional irreducible representation $\pi \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(V)$ can be defined by letting V be the vector space with basis $(v_0, \ldots, v_{n-1})$ such that

- $v_j = \pi(f)^j v_0$ and $\pi(f)v_{n-1} = 0$
- $\pi(e)(v_j) = j(n-j)v_{j-1}$
- $\pi(h)(v_j) = (n - 2j - 1)v_j$

for $j \in \{0, \ldots, n-1\}$, where we set $v_{-1} = 0$.

(a) Show that the 1-dimensional irreducible representation may be given by the zero map $\mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C}$.

(b) Show that the inclusion $\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{C})$ is an irreducible 2-dimensional representation on $V \simeq \mathbb{C}^2$ and relate the basis (v_0, v_1) to the standard basis for $\mathbb{C}^2$.

(c) Show that the adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$ is an irreducible 3-dimensional representation on $V \simeq \mathfrak{sl}_2(\mathbb{C})$ and relate the basis (v_0, v_1, v_2) to the basis (e, f, h).

Exercise 25. *Irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ via homogeneous polynomials.* Let

$$V = \mathrm{Sym}^d(\mathbb{C}^2) := \mathrm{span}_{\mathbb{C}}\{x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d\}$$

be the $(d+1)$-dimensional vector space of homogeneous polynomials of degree d in two variables x, y and define a representation of $\mathfrak{sl}_2(\mathbb{C})$ on V by letting $\pi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(V)$ be determined by

$$\pi(e) = x\frac{\partial}{\partial y}, \qquad \pi(f) = y\frac{\partial}{\partial x}, \qquad \pi(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Show that V is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ and relate the basis $(v_0, \ldots, v_d)$ in Theorem 1.4.5 (see Exercise 24) to the basis $(x^d, \ldots, y^d)$.