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https://home.mathematik.uni-freiburg.de/mathphys/lehre/SoSe21/DiffGeoII.html

Exercise sheet 5

Exercises 20 and 21 make use of the content of the lecture planned for 31 May 2021.

Exercise 19. *Regular elements.* Let $\mathfrak{g}$ denote a finite-dimensional Lie algebra over $\mathbb{C}$ and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Let

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

be the corresponding root space decomposition of Theorem 1.3.3. Give a proof of the following statements.

(a) Let $X \in \mathfrak{h}$. Then the following are equivalent:

- X is regular in $\mathfrak{h}$.
- For all $\alpha \in \mathfrak{h}^* \setminus \{0\}$ we have that $\alpha(X) \neq 0$ whenever $\mathfrak{g}_\alpha \neq \{0\}$.
- $\mathfrak{h} = \ker(\mathrm{ad}_X^{\dim \mathfrak{g}})$.
  Hint. For all $X \in \mathfrak{h}$ show and use that

$$\ker(\mathrm{ad}_X^{\dim \mathfrak{g}}) = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \mathfrak{h}^* \setminus \{0\} \\ \alpha(X)=0}} \mathfrak{g}_\alpha.$$

(b) The set $\{Y \in \mathfrak{g} \mid Y \text{ is regular in } \mathfrak{g}\}$ of regular elements is connected.

Hint. Show that the complement of the zero-set of any polynomial function $P\colon \mathbb{C}^{n+1} \to \mathbb{C}^m$ (i.e. $P = (p_1, \ldots, p_m)^\intercal$ with $p_1, \ldots, p_m \in \mathbb{C}[z_0, \ldots, z_n]$) for $n, m \in \mathbb{N}$ is connected.

Exercise 20. Give a proof of the following statement, which shows that the converse of Theorem 1.3.11 (2) holds:

> **Corollary 1.3.15.** Let $\mathfrak{g}$ denote a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if
>
> $$\mathrm{ad}_\mathfrak{h} = \{\mathrm{ad}_H \mid H \in \mathfrak{h}\}$$
>
> is a maximal Abelian Lie subalgebra of $\mathrm{End}_\mathbb{C}(\mathfrak{g})$ which is simultaneously diagonalizable.

Exercise 21. $\mathfrak{sl}_2$ *triples.* Let $\mathfrak{g}$ denote a semisimple Lie algebra over $\mathbb{C}$ and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra with $\Delta \subset \mathfrak{h}^* \setminus \{0\}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha \neq \{0\}$ for all $\alpha \in \Delta$. Show the following:

(a) Let (h, e, f) be an $\mathfrak{sl}_2$ triple of $\mathfrak{g}$. Then $\mathrm{span}_{\mathbb{C}}\{h, e, f\}$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

(b) For every $\alpha \in \Delta$ set

$$h_\alpha := \frac{2H_\alpha}{\langle \alpha, \alpha \rangle} \qquad e_\alpha := \frac{2E_\alpha}{\langle \alpha, \alpha \rangle} \qquad f_\alpha := E_{-\alpha}$$

where

- $H_\alpha \in \mathfrak{h}$ is such that $\alpha(H) = \kappa(H, H_\alpha)$ for all $H \in \mathfrak{h}$
- $\langle \cdot\,, \cdot \rangle \colon \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ is determined by $\langle \alpha, \beta \rangle = \kappa(H_\alpha, H_\beta)$ for all $\alpha, \beta \in \Delta$
- $E_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ with $[E_\alpha, E_{-\alpha}] = H_\alpha$ and $\kappa(E_\alpha, E_{-\alpha}) = 1$.

Then $(h_\alpha, e_\alpha, f_\alpha)$ is an $\mathfrak{sl}_2$ triple.

(c) For every $\alpha \in \Delta$ and $\beta \in \Delta \cup \{0\}$, the vector space

$$V_\beta := \bigoplus_{\gamma \in \Delta^\alpha_\beta} \mathfrak{g}_\gamma$$

carries a nontrivial representation of the Lie algebra $\mathrm{span}_{\mathbb{C}}\{H_\alpha, E_\alpha, E_{-\alpha}\} \simeq \mathfrak{sl}_2(\mathbb{C})$ via the adjoint action. Here $\Delta^\alpha_\beta := \{\beta + n\alpha \mid n \in \mathbb{Z}\} \cap (\Delta \cup \{0\})$ is the α-string containing β.