Differentialgeometrie II
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https://home.mathematik.uni-freiburg.de/mathphys/lehre/SoSe21/DiffGeoII.html

Exercise sheet 4

Exercise 15. *The radical of* $\mathfrak{g}$. Give a proof of the following proposition (without using Theorem 1.2.19 for part (3)):

> **Proposition 1.2.16.** Let $\mathfrak{g}$ denote a finite dimensional Lie algebra.
>
> (1) $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ is semisimple.
>
> (2) If $\mathfrak{g}$ is semisimple, then the center of $\mathfrak{g}$ is trivial. Moreover, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.
>
> (3) If $\mathfrak{g}$ is simple, then $\mathfrak{g}$ is semisimple.

Exercise 16. *The adjoint representation.* Recall the following theorem.

> **Ado's Theorem.** Let $\mathbb{k}$ be a field of characteristic 0. Then any finite-dimensional Lie algebra over $\mathbb{k}$ is isomorphic to a Lie subalgebra of $\mathrm{Mat}_{n\times n}(\mathbb{k})$ (for some $n \in \mathbb{N}$) whose Lie bracket is given by the commutator of matrices.

Use Proposition 1.2.16 (2) of Exercise 15 to give a proof of this theorem for semi-simple Lie algebras.

Exercise 17. *Cartan subalgebras.* Give a proof of the following statement:

> A nilpotent Lie subalgebra $\mathfrak{h}$ of a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is a Cartan subalgebra if and only if $\mathfrak{h}$ is "self-normalizing", i.e. if and only if $\mathfrak{h} = \mathrm{N}_{\mathfrak{g}}(\mathfrak{h})$, where
>
> $$\mathrm{N}_{\mathfrak{g}}(\mathfrak{h}) := \{X \in \mathfrak{g} \mid \mathrm{ad}_H(X) \in \mathfrak{h} \text{ for all } H \in \mathfrak{h}\}$$
>
> is the *normalizer* of $\mathfrak{h}$ in $\mathfrak{g}$.

Hint. It is always true that $\mathfrak{h} \subset \mathrm{N}_{\mathfrak{g}}(\mathfrak{h}) \subset \mathfrak{g}_0$, where $\mathfrak{g}_0 = \bigcap_{H\in\mathfrak{h}} \ker\big((\mathrm{ad}_H)^{\dim\mathfrak{g}}\big)$. If $\mathfrak{g}_0 \neq \mathfrak{h}$, apply Lie's Theorem 1.2.7 to $\{\mathrm{ad}_H \mid H \in \mathfrak{h}\}$ acting on the $\mathbb{C}$-vector space $\mathfrak{g}_0/\mathfrak{h}$.

Exercise 18. *The root space decomposition for* $\mathfrak{sl}_{n+1}$. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ and let

$$\mathfrak{h} = \{\mathrm{diag}(\lambda_0, \ldots, \lambda_n) \mid \lambda_j \in \mathbb{C}, \lambda_0 = -\textstyle\sum_{j=1}^n \lambda_j\}.$$

Moreover, let $(e_0, \ldots, e_n)$ denote the standard basis of $\mathbb{C}^{n+1}$ and set $E_{ij} = e_i \otimes e_j^*$ for all $i, j \in \{0, \ldots, n\}$. Show the following:

(a) $\mathfrak{h}$ is a Cartan subalgebra.

(b) $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij}$ is the root space decomposition. Moreover, determine the $\alpha_{ij} \in \mathfrak{h}^*$ for which $\mathbb{C}E_{ij} = \mathfrak{g}_{\alpha_{ij}}$ holds.

(c) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathfrak{h}^*$ with $\mathfrak{g}_\alpha \neq \{0\}$, $\mathfrak{g}_\beta \neq \{0\}$ and $\alpha + \beta \neq 0$, while $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$.

(d) $X \in \mathfrak{g}$ is regular if and only if $X \in \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{C})$ with $\operatorname{tr} X = 0$ has $n+1$ distinct eigenvalues.

Hint. Work with the Jordan basis for X and use the Jordan decomposition of X and ad_X.