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Exercise sheet 3

Exercise 11. *Nilpotent Lie algebras.* Let $\mathfrak{g}$ be a nilpotent Lie algebra.

(a) Show that $\mathfrak{g}$ is solvable.

(b) Show that any Lie subalgebra and any quotient Lie algebra of $\mathfrak{g}$ is again nilpotent.

Exercise 12. *Solvable Lie algebras.* Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$.

(a) Show that $\mathfrak{g}$ is solvable if and only if both $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable.

(b) Show that (a) does not hold if "solvable" is replaced by "nilpotent".
Hint. Consider $\mathfrak{g} = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{Mat}_{2\times 2}(\Bbbk) \right\}$ with Lie bracket given by the commutator of matrices.

Exercise 13. *Jordan decomposition.* Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. For $Y \in \mathfrak{g}$, let $\mathrm{ad}_Y = D + N$ denote its Jordan decomposition with D diagonalizable, $N^m = 0$ for some $m \in \mathbb{N}$ and $DN = ND$.

Consider

$$\begin{aligned} \mathrm{ad}^0 \colon \mathrm{End}_{\mathbb{C}}\, \mathfrak{g} &\to \mathrm{End}_{\mathbb{C}}(\mathrm{End}_{\mathbb{C}}\, \mathfrak{g}) \\ F &\mapsto [F, -] \end{aligned}$$

so that

$$(*) \qquad \mathrm{ad}^0_{\mathrm{ad}_Y} = \mathrm{ad}^0_D + \mathrm{ad}^0_N\,.$$

Show that ad^0_D is diagonalizable, that $(\mathrm{ad}^0_N)^{2m} = 0$ and that $[\mathrm{ad}^0_D, \mathrm{ad}^0_N] = 0$, so that $(*)$ is the Jordan decomposition of $\mathrm{ad}^0_{\mathrm{ad}_Y}$.
Hint. For the first claim choose a basis of $\mathfrak{g}$ such that D is given by $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$, consider the corresponding basis E_{ij} of $\mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ and determine $\mathrm{ad}^0_D(E_{ij})$. For the second claim argue as in the proof of Engel's Theorem.

Exercise 14. *The radical of κ.* Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a subfield $\Bbbk$ of $\mathbb{C}$ and let κ be its Killing form. Show that $\mathrm{rad}(\kappa) := \{X \in \mathfrak{g} \mid \kappa(X,Y) = 0,\ \forall Y \in \mathfrak{g}\}$ is

(a) an ideal in $\mathfrak{g}$, and

(b) a solvable Lie subalgebra of $\mathfrak{g}$.

Hint. Determine the Killing form $\overline{\kappa}$ of $\mathrm{rad}(\kappa)$ by observing that, as a $\Bbbk$-vector space, $\mathfrak{g} = \mathrm{rad}(\kappa) \oplus V$ for some vector subspace $V \subset \mathfrak{g}$.