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https://home.mathematik.uni-freiburg.de/mathphys/lehre/SoSe21/DiffGeoII.html

# Exercise sheet 2

**Exercise 6.** *1-parameter subgroups.* Let $G$ be a Lie group and let $\mathfrak{g} = \mathrm{T}_{\mathrm{id}}G$ be its associated Lie algebra. Given $A \in \mathfrak{g}$ let $\gamma\colon \mathbb{R} \to G$ denote the unique smooth curve with $\gamma(0) = \mathrm{id}$ and $\dot\gamma(t) = \gamma(t) \cdot A = A \cdot \gamma(t)$ for all $t \in \mathbb{R}$. Show the following:

(a) $H := \{\gamma(t) \mid t \in \mathbb{R}\}$ is an Abelian Lie subgroup of $G$ of dimension 1 or 0.

(b) Every $X \in \mathfrak{X}^G(G)$ is complete, that is, every integral curve $\gamma\colon (-\epsilon, \epsilon) \to G$ can be extended to an integral curve $\widetilde\gamma\colon \mathbb{R} \to G$, with $\dot{\widetilde\gamma}(t) = X_{\widetilde\gamma(t)}$ for all $t \in \mathbb{R}$.

**Exercise 7.** *Matrix exponentials.* Let $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$.

(a) Let $A, B \in \mathrm{Mat}_{n\times n}(\Bbbk)$. Show that if $AB = BA$, then $\exp(A+B) = \exp(A)\cdot\exp(B)$ holds.

(b) Give an example of $A, B \in \mathrm{Mat}_{n\times n}(\Bbbk)$ such that $\exp(A + B) \neq \exp(A) \cdot \exp(B)$.

(c) Show that for any $A \in \mathrm{Mat}_{n\times n}(\Bbbk)$ one has that $\det(\exp(A)) = \exp(\mathrm{tr}(A))$. In particular, it follows that $\exp\colon\ \mathrm{Mat}_{n\times n}(\Bbbk) \to \mathrm{GL}_n(\Bbbk)$.

**Exercise 8.** *Metrics.* Let $\langle \cdot\,, \cdot \rangle$ denote the standard Euclidean metric on $\mathrm{Mat}_{n\times n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$. Show that the induced metric on $\mathrm{SO}(n)$ is bi-invariant and prove that the 1-parameter subgroups in $\mathrm{SO}(n)$ are geodesics (*without* using Remark 1.1.10).

**Exercise 9.** Give a proof of the following proposition.

**Proposition 1.1.14.** Let $\mathfrak{g}$ denote a finite dimensional Lie algebra over $\Bbbk$.

(1) For every $A \in \mathfrak{g}$, $\mathrm{ad}_A$ is a *derivation* on $\mathfrak{g}$, that is, an endomorphism of $\mathfrak{g}$ such that for all $B, C \in \mathfrak{g}$ we have

$$\mathrm{ad}_A([B, C]) = [\mathrm{ad}_A(B), C] + [B, \mathrm{ad}_A(C)].$$

Moreover, $\mathrm{ad}_{[B,C]} = [\mathrm{ad}_B, \mathrm{ad}_C]$.

(2) The Killing form $\kappa$ on $\mathfrak{g}$ is a symmetric bilinear form $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to \Bbbk$ which obeys

$$\kappa(\mathrm{ad}_A(B), C) = -\kappa(B, \mathrm{ad}_A(C))$$

for all $A, B, C \in \mathfrak{g}$.

**Exercise 10.** *Complexification and real forms.* Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{k}$.

(a) Prove that $[\mathfrak{g},\mathfrak{g}] := \mathrm{span}_{\mathbb{k}}\{[X,Y] \mid X,Y \in \mathfrak{g}\}$ is an ideal of $\mathfrak{g}$.

(b) If $\mathbb{k} = \mathbb{R}$ show that $[\mathfrak{g}^{\mathbb{C}},\mathfrak{g}^{\mathbb{C}}] \simeq [\mathfrak{g},\mathfrak{g}]^{\mathbb{C}}$ as Lie algebras, where $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ denotes the complexification of an $\mathbb{R}$-vector space $V$.

(c) Show that $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{su}(n)$ are both real forms of the complex Lie algebra $\mathfrak{sl}_n(\mathbb{C})$.