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Exercise sheet 11

Exercise 43. *Nilpotent cone and Slodowy slice.*

(a) Show that the set

$$\mathrm{Nil}(\mathfrak{sl}_2(\mathbb{C})) = \{A \in \mathfrak{sl}_2(\mathbb{C}) \mid A^n = 0 \text{ for some } n > 0\}$$

of nilpotent matrices in $\mathfrak{sl}_2(\mathbb{C})$ naturally admits the structure of an affine algebraic variety isomorphic to $X_1 = (\mathbb{C}^2/\mathbb{Z}_2, \mathbb{C}[z_1, z_2]^{\mathbb{Z}_2})$.

(b) Consider

$$x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad y = x^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

with $x, y \in \mathfrak{sl}_3(\mathbb{C})$ and let $\mathcal{S} := x + \ker \mathrm{ad}_y$, $\mathcal{S} \subset \mathfrak{sl}_3(\mathbb{C})$. Moreover, let

$$\begin{aligned} \chi\colon \mathcal{S} &\to \mathbb{A}^2_{\mathbb{C}} \\ A &\mapsto (\sigma_3(A), \sigma_2(A)) \end{aligned}$$

with $\sigma_3(A) = \det A$ and $\sigma_2(A) = \sum_{j=1}^3 \det A_{jj}$, where A_{jj} is the matrix obtained from A by deleting the jth row and column.

Show that $X_2 = \chi^{-1}(0)$ naturally admits the structure of an affine algebraic variety isomorphic to $(\mathbb{C}^2/\mathbb{Z}_3, \mathbb{C}[z_1, z_2]^{\mathbb{Z}_3})$.

Note. In part (a), $\mathrm{Nil}(\mathfrak{sl}_2(\mathbb{C})) \simeq X_1$ arises similarly with $x = y^T = 0$ and $\chi = \det$.

Exercise 44. (a) Let $S = \{q \in \mathbb{H} \mid \overline{q} \cdot q = 1\}$ be the multiplicative group of unit quaternions. For any $q \in S$ write $q = a + b\,\mathrm{j}$ with $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. Show that

$$\begin{aligned} S &\to \mathrm{SU}(2) \\ a + b\,\mathrm{j} &\mapsto \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \end{aligned}$$

is a group isomorphism.

(b) (1) Let $V = \{q \in \mathbb{H} \mid \overline{q} = -q\} \simeq \mathbb{R}^3$ and $\rho(q)(v) := q \cdot v \cdot \overline{q}$ for all $q \in S$ and all $v \in V$. Show that this yields a group homomorphism $\rho\colon S \to \mathrm{End}_{\mathbb{R}}(V)$.

(2) With ρ as in (1), show that

$$\begin{aligned} \ker(\rho) &= \{\pm 1\} \\ \operatorname{im}(\rho) &= \{A \in \operatorname{End}_{\mathbb{R}}(V) \mid \det(A) = 1 \text{ and } \langle Av, Av\rangle = \langle v, v\rangle \text{ for all } v \in V\}. \end{aligned}$$

Since $V \simeq \mathbb{R}^3$, it follows that $\operatorname{im}(\rho) \simeq \mathrm{SO}(3)$ and since $S \simeq \mathrm{SU}(2)$ one has a group homomorphism $\phi\colon \mathrm{SU}(2) \xrightarrow{2:1} \mathrm{SO}(3)$.

Hint. To show that $\operatorname{im}(\rho) \subset \mathrm{SO}(3)$, note that $\operatorname{im}(\rho)$ is connected and preserves the Euclidean norm on V. Thus ρ induces a continuous group homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. To prove surjectivity, use either that $\mathrm{SU}(2)$ is simple and compact, or argue that an open neighbourhood of the identity in $\mathrm{SU}(2)$ is mapped to an open neighbourhood of the identity in $\mathrm{SO}(3)$.

Exercise 45. Let $\phi\colon \mathrm{SU}(2) \xrightarrow{2:1} \mathrm{SO}(3)$ be as in Exercise 44 and assume that the isomorphism $V \simeq \mathbb{R}^3$ maps the basis $(\mathrm{i}, \mathrm{j}, \mathrm{k})$ to the standard basis of $\mathbb{R}^3$. Show that

(a) $\phi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & -\sin 2\theta & \cos 2\theta \end{pmatrix}$ where $\lambda = \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C}$ for some $\theta \in \mathbb{R}$

(b) $\phi\left(\pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Exercise 46. *Automorphisms of Coxeter–Dynkin diagrams.* An *automorphism* of a Coxeter–Dynkin diagram is a permutation σ of the nodes of the diagram such that two nodes are joined by an edge if and only if their images under σ are joined by the same type of edge, respecting the orientation in the sense that if nodes i and j are joined by a multiple edge with oriented towards j, then the multiple edge between $\sigma(i)$ and $\sigma(j)$ is oriented towards $\sigma(j)$.

Determine the full automorphism groups of the Coxeter–Dynkin diagrams of types A_n, D_n, E_6, E_7, E_8.