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https://home.mathematik.uni-freiburg.de/mathphys/lehre/SoSe21/DiffGeoII.html

Exercise sheet 10

Exercise 39. *Holomorphic volume forms.* Let M be a compact complex manifold with trivial canonical bundle K_M. Show that for any two holomorphic volume forms $\eta, \widetilde{\eta}$, there exists some $\lambda \in \mathbb{C}^*$ such that $\eta = \lambda\widetilde{\eta}$.

Exercise 40. *The A_1 singularity.*

(a) Let $E := \pi^{-1}(0)$ be the exceptional divisor in the resolution $\pi\colon \widetilde{X}_1 \to X_1$ of the singularity $X_1 \subset \mathbb{A}^3_{\mathbb{C}}$ with $I(X_1) = \langle x_0^2 - x_1x_2 \rangle \subset \mathbb{C}[x_0, x_1, x_2]$ as in Example 2.1.13. Prove that $E \simeq \mathbb{CP}^1$.

(b) Show that the section of $K_{\widetilde{X}_1}$ constructed chart by chart in Example 2.1.13 descends from a multiple of $\mathrm{d}z_1 \wedge \mathrm{d}z_2$ on $\mathbb{C}^2$ when viewing X_1 as the quotient singularity $X_1 \simeq \mathbb{C}^2/\mathbb{Z}_2$ as in Example 2.1.6.

Exercise 41. *Resolution of submanifolds.* Let $U \subset \mathbb{C}^N$ be an n-dimensional complex submanifold of $\mathbb{C}^N$.

(a) Let $P \subset U$ be a submanifold of U. Show that the construction of the resolution of U along P given in Definition 2.1.14 (1) is independent of the choice of adapted coordinates.

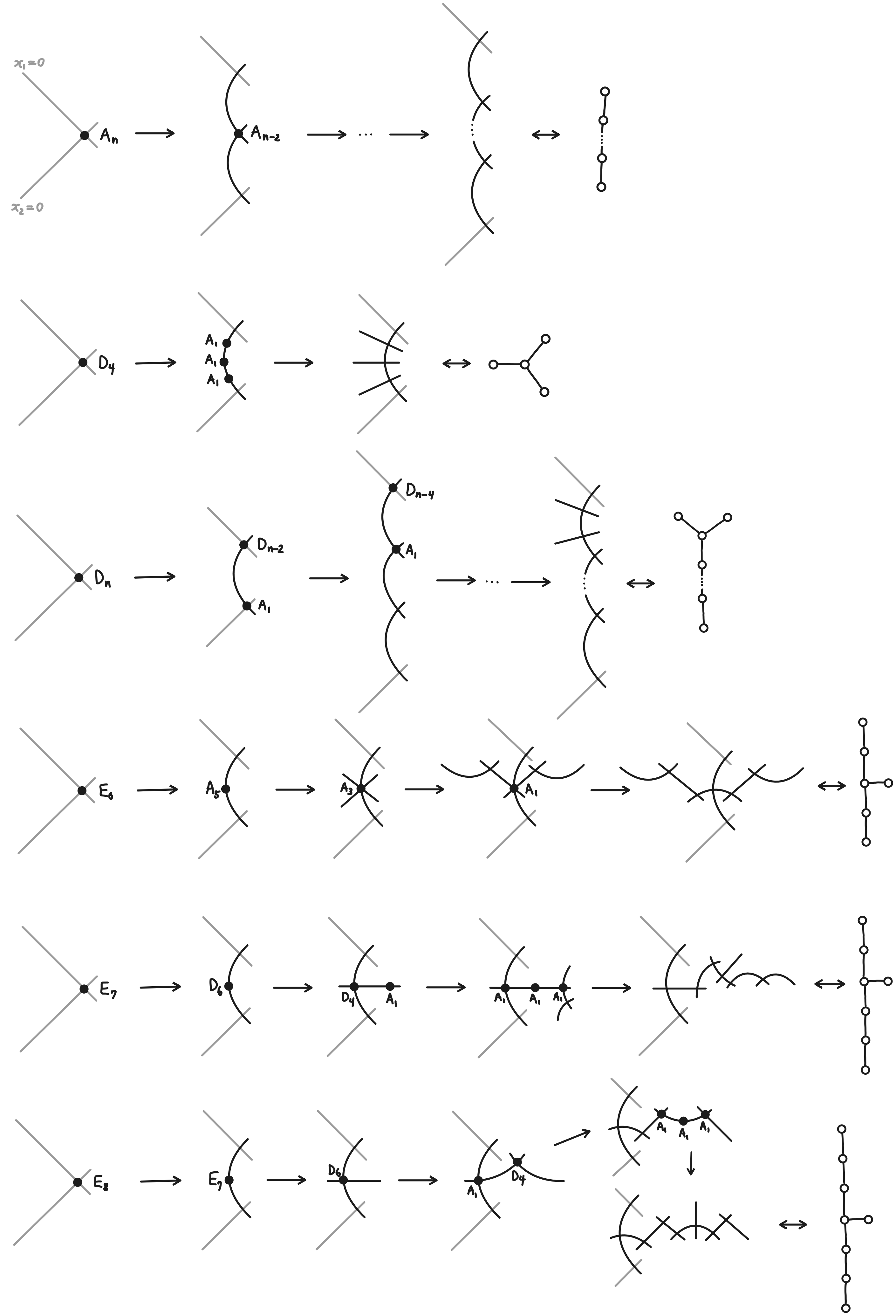
This allows us to define $\sigma\colon \widetilde{U} \to U$ a resolution of U along P, where $\widetilde{U}_V = \sigma^{-1}(V)$ for all open coordinate neighbourhoods $V \subset U$ yield a holomorphic atlas for $\widetilde{U}$. Calculate $\sigma^{-1}(z)$ for any $z \in U$.

(b) Let

- $f \in \mathbb{C}[z_1, \ldots, z^N] \setminus \{0\}$
- $X = \{z \in U \mid f(z) = 0\}$
- $p \in X$ and
- $\widetilde{X}$ the blow-up of X in p as in Definition 2.1.14 (2).

Prove that $\widetilde{X}$ is isomorphic to the resolution of X in p if $X \subset U \subset \mathbb{C}^N$ is a smooth submanifold.

Exercise 42. *Du Val singularities.* Prove that the exceptional divisors of the resolutions of any one of the types A_n, D_n, E_6, E_7, E_8 of the du Val singularities arise under repeated blow-ups, yielding the picture on the following page. Show that these resolutions are crepant.


$x_1=0$
$x_2=0$
A_n
A_{n-2}
D_4
A_1
A_1
A_1
D_n
D_{n-2}
A_1
D_{n-4}
A_1
E_6
A_5
A_3
A_1
E_7
D_6
D_4
A_1
A_1
A_1
A_1
E_8
E_7
$D6$
A_1
$D4$
A_1
A_1
A_1