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https://home.mathematik.uni-freiburg.de/mathphys/lehre/SoSe21/DiffGeoII.html

Exercise sheet 1

Exercise 1.

(a) Let $F\colon M \to N$ be a diffeomorphism between smooth manifolds M and N and let $X, Y \in \mathfrak{X}(M)$ be vector fields on M. Show that

$$DF([X,Y]) = [DF(X), DF(Y)]$$

where for all $p \in M$, $(DF(X))_{F(p)} = DF_p(X_p)$.

(b) Let G be a Lie group and let $X \in \mathfrak{X}^G(G)$ be a left-invariant vector field on G. Show that

$$(ab) \cdot X_c = a \cdot (b \cdot X_c)$$

for all $a, b, c \in G$, where $a \cdot X_b := (DL_a)_b(X_b)$.

Exercise 2. *Lie algebras associated to Lie groups.* Let G be a Lie group. Show that the Lie algebra associated to G, defined as $\mathfrak{g} := \mathrm{T}_{\mathrm{id}}G$ with Lie bracket induced by the commutator of vector fields via the isomorphism $\mathfrak{X}^G(G) \simeq \mathrm{T}_{\mathrm{id}}G$, is indeed a Lie algebra.

Exercise 3. Prove that the groups

$$\mathrm{SO}(n),\ \mathrm{Sp}_{\frac{n}{2}}(\mathbb{R}),\ \mathrm{U}(n),\ \mathrm{SU}(n),\ \mathrm{SL}_n(\mathbb{R}),\ \mathrm{SL}_n(\mathbb{C})$$

are Lie subgroups of $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$ for at least two of these groups. (Here $\mathrm{Sp}_{\frac{n}{2}}$ is only defined for even n.) Determine their dimension and their tangent space at the identity and prove that the tangent space at the identity is closed under the commutator of matrices.

Exercise 4. *Matrix Lie groups as closed submanifolds.* Recall that a *matrix Lie group* is a Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ which is closed in $\mathrm{GL}_n(\mathbb{C})$. Prove or disprove the following claim:

> A matrix Lie group $G \subset \mathrm{GL}_n(\mathbb{C}) \subset \mathrm{Mat}_{\mathbb{C}}(n \times n)$ is also a closed submanifold of $\mathrm{Mat}_{\mathbb{C}}(n \times n)$.

Exercise 5. *Low-dimensional Lie algebras.*

(a) Show that up to isomorphism there is a unique non-Abelian Lie algebra of dimension 2.

(b) Give at least three (mutually non-isomorphic) examples of non-Abelian Lie algebras of dimension 3.