We can rearrange ① to give λ_1 in terms of x, y, z and λ_2. Doing this, we substitute for λ_1 in equations ② and ③. Now, we rearrange equation ② for λ_2 and substitute for λ_2, in equation ③. Equation ③ is now in terms of x, y and z alone.

Rearranging ③ can give z in terms of x and y, so we use this to substitute for z in ⑤ and ④. Then we rearrange ④ to give y in terms of x and substitute this expression in to equation ⑤. We can now solve ⑤ for x. Using the expression for y in terms of x and z in terms of x and y, we can then recover the other two coordinates, to give use a complete answer.

Question 1) Find the maximum value of $F(x,y) = 49 - x^2 - y^2$, on the line $x + 3y = 10$

2) Find the point on the plane $x + 2y + 3z = 13$, closest to the point $(1,1,1)$

3) Suppose that the temperature at point (x,y,z) on the sphere $x^2+y^2+z^2=1$ is $T = 400xyz^2$. Find the lowest and highest temperatures on the sphere.

4) Maximise the function $F(x,y,z) = x^2 - 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

Solution

1) $\nabla F = \lambda \nabla(x + 3y - 10)$ gives us

$\frac{\partial F}{\partial x}\underline{i} = \lambda \underline{i}$, $\frac{\partial F}{\partial y}\underline{j} = \lambda 3\underline{j}$. $\frac{\partial F}{\partial x} = -2x$, $\frac{\partial F}{\partial y} = -2y$.

Losing the $\underline{i}$ and $\underline{j}$ we have that

① $-2x = \lambda$

② $-2y = 3\lambda$

③ $x + 3y - 10 = 0$.

Substituting for λ in equation ② gives $-2y = -6x$, which reduces to $y = 3x$.

Substituting for y in equation ③ gives $x + 9x - 10 = 0$, which simplifies to $10x = 10$ or $x = 1$. We know that $y = 3x$, so if $x = 1$, $y = 3$.

The question asks for the value of F at its maximum.

$F(1,3) = 49 - (1)^2 - (3)^2 = 39$.

2) Firstly, we must ~~for~~ construct an equation to minimize. Let's try to minimize the distance from the point $(1,1,1)$ to the point (x,y,z).