Tutorial 8

Imagine that you have an equation that tells you the temperature at every point in a room. You could use the maxima, minima and saddle point methods, that we looked at earlier in the course, to find the point, in the room, with the lowest or highest temperature. Now, lets put a large sphere in the room. How would we calculate the point on the sphere with the highest temperature? The answer is that we use Lagrange multipliers.

To use Lagrange multipliers, we firstly need to turn the question into an equation to maximize or minimize and a constraint. In this example, we want to maximize the equation for the temperature of the points in the room, subject to the constraint that the points must lie on the sphere.

If T is the equation of temperature of the room and g the equation of a sphere $(x^2+y^2+z^2-1=0)$, then we set up our equations as follows.

$\nabla T = \lambda \nabla g$ where λ is the Lagrange multiplier.

The definition of ∇T is ~~∂T~~ $\nabla T = \frac{\partial T}{\partial x}\underset{\sim}{i} + \frac{\partial T}{\partial y}\underset{\sim}{j} + \frac{\partial T}{\partial z}\underset{\sim}{k}$.

So our equation becomes.

$$\frac{\partial T}{\partial x}\underset{\sim}{i} + \frac{\partial T}{\partial y}\underset{\sim}{j} + \frac{\partial T}{\partial z}\underset{\sim}{k} = \lambda\frac{\partial g}{\partial x}\underset{\sim}{i} + \lambda\frac{\partial g}{\partial y}\underset{\sim}{j} + \lambda\frac{\partial g}{\partial z}\underset{\sim}{k}.$$

where $\underset{\sim}{i}$ is a vector of length 1 along the x-axis, $\underset{\sim}{j}$ a vector of length 1 along the y axis and $\underset{\sim}{k}$ a vector along the z-axis of length 1. Because a vector along the x axis can't be made up from vectors along the y or z axes, we can equate the $\underset{\sim}{i}$ terms on both sides. Likewise with the $\underset{\sim}{j}$ and $\underset{\sim}{k}$.

So
$$\frac{\partial T}{\partial x}\underset{\sim}{i} = \lambda\frac{\partial g}{\partial x}\underset{\sim}{i}$$

$$\frac{\partial T}{\partial y}\underset{\sim}{j} = \lambda\frac{\partial g}{\partial y}\underset{\sim}{j}$$

$$\frac{\partial T}{\partial z}\underset{\sim}{k} = \lambda\frac{\partial g}{\partial z}\underset{\sim}{k}$$

Now, we can lose the $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ bits to give

$$\frac{\partial T}{\partial x} = \lambda\frac{\partial g}{\partial x}, \quad \frac{\partial T}{\partial y} = \lambda\frac{\partial g}{\partial y}, \quad \frac{\partial T}{\partial z} = \lambda\frac{\partial g}{\partial z}$$