To find the point where both gradients are zero (ie to find the maxima, minima or saddle points) first differentiate the equation with respect to x and set it equal to zero. Then take the original equation and differentiate it with respect to y and set it equal to zero. Solve the two equations simultaneously for the values of x & y.

So, once we have found a point with zero gradients, how do we work out if it's a maximum, a minimum or a saddle point?

The answer is that we work out $\left(\frac{\partial^2 F}{\partial x^2}\right)\left(\frac{\partial^2 F}{\partial y^2}\right) - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2$ and substitute in the values of x & y.

If the answer is greater than zero, the point is either a maximum or a minimum. If it is less than zero, the point is a saddle point.

If it turns out that the point is a maximum or a minimum, then to work out which it is, we look at the value of $\frac{\partial^2 F}{\partial x^2}$. If $\frac{\partial^2 F}{\partial x^2}$ is positive, then it's a minimum. If $\frac{\partial^2 F}{\partial x^2}$ is negative, then it's a maximum.

Question 3) Find the maxima, minima and saddle points of

I) $F(x,y) = x^2 + 2xy$ II) $x^3 - y^3 - 2xy + 6$ III) $3y^2 - 2y^3 - 3x^2 + 6xy$

Solutions

I) $F(x,y) = x^2 + 2xy$ $\frac{\partial F}{\partial x} = 2x + 2y$ $\frac{\partial F}{\partial y} = 2x$

$2x + 2y = 0$

$2x = 0 \Rightarrow x = 0$ substitute into $2x + 2y = 0$, $2y = 0 \Rightarrow y = 0$.

the point $(0,0)$ is either a maximum, minimum or saddle point.

$\frac{\partial^2 F}{\partial x^2} = 2$ $\frac{\partial^2 F}{\partial y^2} = 0$ $\frac{\partial^2 F}{\partial x \partial y} = 2$ $\left(\frac{\partial^2 F}{\partial x^2}\right)\left(\frac{\partial^2 F}{\partial y^2}\right) - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 = -4$

$\therefore (0,0)$ is a saddle point

II) $F(x,y) = x^3 - y^3 - 2xy + 6$ $\frac{\partial F}{\partial x} = 3x^2 - 2y$ $\frac{\partial F}{\partial y} = -3y^2 - 2x$

$3x^2 - 2y = 0$

$-3y^2 - 2x = 0 \Rightarrow 2x = -3y^2 \Rightarrow x = -\frac{3}{2}y^2$. substitute into $3x^2 - 2y = 0$

$\frac{27}{4}y^4 - 2y = 0 \Rightarrow y\left(\frac{27}{4}y^3 - 2\right) = 0$. Either $y = 0$ or $\frac{27}{4}y^3 - 2 = 0$

$\frac{27}{4}y^3 = 2$, $y^3 = \frac{8}{27}$, $y = \frac{2}{3}$.

Either $y = 0$ or $y = \frac{2}{3}$. Now $x = -\frac{3}{2}y^2$. So if $y = 0$, $x = 0$. If $y = \frac{2}{3}$, then $x = -\frac{3}{2}\left(\frac{2}{3}\right)^2 = -\frac{2}{3}$. Our two points are $(0,0)$ and $\left(-\frac{2}{3}, \frac{2}{3}\right)$

$\frac{\partial^2 F}{\partial x^2} = 6x$ $\frac{\partial^2 F}{\partial y^2} = -6y$ $\frac{\partial^2 F}{\partial x \partial y} = -2$

At $(0,0)$ $\left(\frac{\partial^2 F}{\partial x^2}\right)\left(\frac{\partial^2 F}{\partial y^2}\right) - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 = -4$ $\therefore (0,0)$ is a saddle point.