Often, a question will ask for the taylor expansion of a function upto a certain order. The order indicates how much of the series you have to work out. Second order would mean include all the terms that have no, single or double differentiation, but not triple or quadrouple differentiation. Third order means include all terms with triple differentiation or less, but not the terms with more.

Just to clear up any confusion, a Taylor expansion is exactly the same thing as a Taylor series. The two names tend to be used interchangeably.

Question 1) Find the Taylor series in powers of x & y of the functions I) xe^y II) $y\sin x$ III) $\cos(x^2+y^2)$ upto 3rd order.

Solution

I) There are two ways to tackle this. Firstly, we can use the full expression that we saw for $F(x,y)$. However, because the expression contains only one factor of x, w we differentiate it twice or more, with respect to x, it will be zero. So instead, we can factor out the x and just expand the factor e^y.

The first way: $F = xe^y$ $F(0,0)=0$, $\frac{\partial F}{\partial x} = e^y$, $\left.\frac{\partial F}{\partial x}\right|_{\substack{x=0\\y=0}} = 1$

$$\frac{\partial F}{\partial y} = xe^y,\ \left.\frac{\partial F}{\partial y}\right|_{\substack{x=0\\y=0}} = 0,\ \frac{\partial^2 F}{\partial x \partial y} = e^y,\ \left.\frac{\partial^2 F}{\partial x \partial y}\right|_{\substack{x=0\\y=0}} = 1,\ \frac{\partial^2 F}{\partial x^2} = 0,\ \left.\frac{\partial^2 F}{\partial y^2}\right|_{\substack{x=0\\y=0}} = 0$$

$$\frac{\partial^3 F}{\partial x^3} = 0,\ \frac{\partial^3 F}{\partial y^3} = xe^y,\ \left.\frac{\partial^3 F}{\partial y^3}\right|_{\substack{x=0\\y=0}} = 0,\ \frac{\partial^3 F}{\partial^2 x \partial y} = 0,\ \frac{\partial^3 F}{\partial x \partial^2 y} = e^y,\ \left.\frac{\partial^3 F}{\partial x \partial y^2}\right|_{\substack{x=0\\y=0}} = 1$$

$$\text{so } F(x,y) = 0 + x + 0 + 0 + xy + 0 + 0 + 0 + \frac{3}{3!}xy^2 + 0$$
$$= x + xy + \frac{1}{2!}xy^2 + \ldots$$

The second way: $F = xe^y = x(e^y)$. We expand e^y in one variable

$$\left.\frac{de^y}{dy}\right|_{y=0} = 1,\ \left.e^y\right|_{y=0} = 1,\ \left.\frac{d^2e^y}{dy^2}\right|_{y=0} = 1,\ \left.\frac{d^3e^y}{dy^3}\right|_{y=0} = 1$$

$$xe^y = x\left(1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \ldots\right) = x + xy + \frac{1}{2!}xy^2 + \frac{1}{3!}xy^3 + \ldots$$

Please note that the two ways are in agreement up to xy^2, but the first method hasn't uncovered as many of the terms as the second. If we worked to infinitely high order, the two methods would give the same answer.

II) First way: $F = y\sin x$ $F(0,0)=0$, $\left.\frac{\partial F}{\partial y}\right|_{\substack{x=0\\y=0}} = 0$, $\left.\frac{\partial F}{\partial x}\right|_{\substack{x=0\\y=0}} = 0$,

$$\frac{\partial^2 F}{\partial y^2} = 0,\ \left.\frac{\partial^2 F}{\partial x^2}\right|_{\substack{x=0\\y=0}} = 0,\ \left.\frac{\partial^2 F}{\partial x \partial y}\right|_{\substack{x=0\\y=0}} = 1,\ \left.\frac{\partial^3 F}{\partial x^3}\right|_{\substack{x=0\\y=0}} = 0,\ \left.\frac{\partial^3 F}{\partial y^3}\right|_{\substack{x=0\\y=0}} = 0,\ \frac{\partial^3 F}{\partial x \partial y^2} = 0,\ \left.\frac{\partial^3 F}{\partial x^2 \partial y}\right|_{\substack{x=0\\y=0}} = 0$$