The substitution also effects the Formula for the coefficients a_n, a_0 and b_n.

For the interval $[-L,L]$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Question 1

Calculate the Fourier series of the functions

I) $f(x) = x$ in $[-L,L]$

II) $f(x) = x^2$ in $[-\pi,\pi]$

Draw a graph of the solution to (I) in the interval $[-3L,3L]$ and the solution to (II) in the interval $[-3\pi,3\pi]$.

Solution

I) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{-L}^{L} x\,dx = \frac{1}{L}\left[\frac{1}{2}x^2\right]_{-L}^{L} = \frac{1}{L}\left(\frac{1}{2}L^2 - \frac{1}{2}L^2\right) = 0$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x\cos\frac{k\pi x}{L}\,dx = \frac{1}{L}\left(\left[\frac{xL}{\pi k}\sin\frac{k\pi x}{L}\right]_{-L}^{L} - \int_{-L}^{L}\frac{L}{\pi k}\sin\frac{k\pi x}{L}\,dx\right) = \frac{1}{L}\left(\left(\frac{L^2}{\pi k}\sin k\pi + \frac{L}{\pi k}\sin(-k\pi)\right) + \left[\frac{L^2}{\pi^2 k^2}\cos\frac{k\pi x}{L}\right]_{-L}^{L}\right)$$

$$= \frac{1}{L}\left(\frac{L^2}{\pi^2 k^2}\cos k\pi - \frac{L^2}{\pi^2 k^2}\cos(-k\pi)\right) = 0$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{L} x\sin\frac{k\pi x}{L}\,dx = \frac{1}{L}\left(\left[\frac{-xL}{k\pi}\cos\frac{k\pi x}{L}\right]_{-L}^{L} - \int_{-L}^{L}\frac{-L}{k\pi}\cos\frac{k\pi x}{L}\,dx\right) = \frac{1}{L}\left(-\frac{L^2}{k\pi}\cos(k\pi) - \frac{L}{k\pi}\cos(-k\pi) + \left[\frac{L^2}{k^2\pi^2}\sin\frac{k\pi x}{L}\right]_{-L}^{L}\right)$$

$$= \frac{1}{L}\left(\frac{-2L^2}{k\pi}\cos(k\pi) + \frac{L^2}{k^2\pi^2}\sin(k\pi) - \frac{L^2}{k^2\pi^2}\sin(-k\pi)\right) = \frac{1}{L}\left(-\frac{2L^2}{k\pi}\cos(k\pi)\right)$$

$$= -\frac{2L}{k\pi}(-1)^k = \frac{2L}{k\pi}(-1)^{k+1}$$

So $$f(x) = \sum_{n=1}^{\infty}\frac{2L}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L}$$