When we calculate a_k and b_k, if we leave k as a variable, ie we don't give k a specific value, then the expression that we obtain for a_k and b_k is a formula containing k. We can substitute these formulas into the expression $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ to obtain an exact, infinite series.

The reason that Fourier analysis is useful is that it enables us to describe a function with the a_k and b_k coefficients and this tells us which wavelength play a big part in making up the function and which wavelengths play a small part.

For example, suppose we were studying a fluid mechanics problem (or an AC circuit problem or an acoustics or resonance problem) and that we had managed to work out that there was an undesirable and complicated wave that kept damaging the system. If we knew the shape of the wave, then we could use Fourier analysis to see which frequencies of sin and cos waves contributed most greatly to the complicated wave. We could then adapt the system so that it wasnt susceptible to these frequencies.

Question

Calculate the Fourier series of the following functions, writing out the 'general' term.

I) $F(x) = \begin{cases} 1 & -\pi < x < 0 \\ -1 & 0 < x < \pi \end{cases}$

II) $F(x) = |x| \qquad -\pi < x < \pi$

Draw graphs of these series in the interval $[-3\pi, 3\pi]$

Solution

I) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} dx + \frac{1}{\pi}\int_{0}^{\pi} -1\,dx = \frac{1}{\pi}\left[x\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[-x\right]_{0}^{\pi} = 1 - 1 = 0 \qquad a_0 = 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\cos kx\,dx + \frac{1}{\pi}\int_{0}^{\pi} -\cos kx\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin kx}{k}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{\sin kx}{k}\right]_{0}^{\pi} = \frac{1}{\pi}\left(\frac{\sin(0)}{k} - \frac{\sin(-k\pi)}{k} - \frac{\sin(k\pi)}{k}\right.$$

$$\left. + \frac{\sin(0)}{k}\right). \qquad \sin 0 = 0, \quad \sin k\pi = 0, \quad \text{so } a_k = 0$$