$$\int_{-\pi}^{\pi} F(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx = \frac{a_0}{2}\left[x\right]_{-\pi}^{\pi} = \tfrac{1}{2}a_0\,(2\pi) = a_0\pi.$$

Hence, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx.$

Now let's look at $\int_{-\pi}^{\pi} F(x)\cos kx\,dx$

$$\int_{-\pi}^{\pi} F(x)\cos kx\,dx = \int_{-\pi}^{\pi}\frac{a_0}{2}\cos kx\,dx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\cos kx\,dx + \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\sin nx\cos kx\,dx$$

We've already said that the integral of any $\cos kx$ function is zero, so the first term on the right hand side is zero.

The integral of the product of $\sin nx$ and $\cos kx$ is also zero because, the area above and below the curve is the same for any value of k and n. Hence, the third term on the right hand side is also zero.

The same is also true of the integral of the product of two cosine functions, except when n is equal to k. When $n=k$, the integral is of $\cos^2 kx$ which is positive or zero everywhere, so the area under the function $\cos^2 kx$ is positive.

The entire right hand side reduces to $a_k\int_{-\pi}^{\pi}\cos^2 kx\,dx$.

Solving this, we use the trig identity $\cos^2 kx = \frac{1}{2}(1+\cos 2kx)$

$$\int_{-\pi}^{\pi} F(x)\cos kx\,dx = a_k\int_{-\pi}^{\pi}\cos^2 kx\,dx = a_k\int_{-\pi}^{\pi}\tfrac{1}{2}(1+\cos 2kx)\,dx$$

$$= \frac{a_k}{2}\left[x\right]_{-\pi}^{\pi} + \frac{a_k}{2}\left[\frac{1}{2k}\sin 2kx\right]_{-\pi}^{\pi} = \pi a_k$$

Hence, $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos kx\,dx.$

We can use similar logic to obtain the coefficients b_k

$$\int_{-\pi}^{\pi} F(x)\sin kx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\sin kx\,dx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\sin kx\,dx + \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\sin nx\sin kx$$

In this expression, every term is zero, except the b_k term, because the b_k term contains $\sin^2 kx$, which has a nonzero integral.

$$\int_{-\pi}^{\pi} F(x)\sin kx\,dx = b_k\int_{-\pi}^{\pi}\sin^2 kx\,dx = b_k\int_{-\pi}^{\pi}\tfrac{1}{2}(1-\cos 2kx)\,dx,$$ using the trig identity $\sin^2 kx = \frac{1}{2}(1-\cos 2kx)$

$$b_k\int_{-\pi}^{\pi}\tfrac{1}{2}(1-\cos 2kx)\,dx = \frac{b_k}{2}\left[x\right]_{-\pi}^{\pi} - \frac{b_k}{2}\left[\frac{1}{2k}\sin 2kx\right]_{-\pi}^{\pi} = b_k\pi$$

So $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin kx\,dx$