$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix}$$

This gives two equations:-

$2a + b = 3a$ From the top row and $a + 2b = 3b$ From the bottom row. Rearranging the Former gives $b = a$. Rearranging the latter also gives $a = b$.

When calculating eigenvectors, we often can't calculate the components exactly but obtain only relationships between components. Here, we have obtained that $a = b$.

Given this relationship, we are free to choose one of the components and then use the relationship to obtain the others. Choosing $b = 1$, $a = b$ tells us that $a = 1$. Hence, the eigenvector corresponding to $\lambda = 3$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Now we try for the other eigenvector. If $\lambda = 1$, then:-

$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ This gives two equations

$2a + b = a$
$a + 2b = b$. Rearranging the upper equation gives $a = -b$.

Similarly, rearranging the lower one gives $a = -b$. We have only obtained one relationship, so we are free to choose one of the components and use the relationship to calculate the other. Choosing $b = 1$, $a = -b$, so $a = -1$.

The eigenvector corresponding to $\lambda = 1$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

<u>Note</u>

To avoid confusion, I should explain that solving $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$ for a and b is exactly the same as solving $\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$ because $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$ is the same as $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$. Subtracting the right hand side gives $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$ or $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$