Solution

a) For $\underline{u}$ perpendicular to $\underline{v}$, $\underline{u}\cdot\underline{v}=0$, so the question is what values of k make $\underline{u}\cdot\underline{v}=0$.

$$\begin{pmatrix}2\\1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\7\\k\end{pmatrix} = 2+7+3k \qquad 2+7+3k=0$$

$$3k=-9$$

$$k=-3$$

b) $$\begin{pmatrix}k\\k\\1\end{pmatrix}\cdot\begin{pmatrix}k\\5\\6\end{pmatrix} = k^2+5k+6 \qquad k^2+5k+6=0$$

$$(k+3)(k+2)=0$$

$$k=-3 \text{ or } -2$$

Question 3

Use Cauchy-Schwartz to prove that, for all values of θ,

$$(a\cos\theta + b\sin\theta)^2 \leq a^2+b^2$$

Solution

We start with 2 vectors $\begin{pmatrix}a\\b\end{pmatrix}$ & $\begin{pmatrix}c\\d\end{pmatrix}$. Applying the Cauchy-Schwartz inequality

$$\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}c\\d\end{pmatrix} \leq \left\|\begin{pmatrix}a\\b\end{pmatrix}\right\| \left\|\begin{pmatrix}c\\d\end{pmatrix}\right\|$$

The length of the vector, denoted $\left\|\begin{pmatrix}a\\b\end{pmatrix}\right\|$, is given by $\sqrt{a^2+b^2}$ because of Pythagoras' rule. So, substituting for $\left\|\begin{pmatrix}a\\b\end{pmatrix}\right\|$ and $\left\|\begin{pmatrix}c\\d\end{pmatrix}\right\|$

$$ac+bd \leq \left(\sqrt{a^2+b^2}\right)\left(\sqrt{c^2+d^2}\right).$$

Squaring this,

$$(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$$

Lets take $c=\cos\theta$ and $d=\sin\theta$

$$(a\cos\theta + b\sin\theta)^2 \leq (a^2+b^2)(\cos^2\theta+\sin^2\theta)$$

However $\cos\theta + \sin^2\theta = 1$, so

$$(a\cos\theta + b\sin\theta)^2 \leq (a^2+b^2) \quad \text{as required.}$$

Question 4

Prove that the set $\{1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta\}$ is orthogonal in the interval $[-\pi, \pi]$ and make it orthonormal.

Solution

Firstly, let's take the product between all pairs of functions in the set. This will demonstrate the orthogonality.