In 3 dimensions $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$.

To get $\underline{u} \cdot \underline{v}$, we multiply the components of $\underline{u}$ & $\underline{v}$ together and then we sum over the components, ie $\underline{u} \cdot \underline{v} = \sum_i u_i v_i$. Replacing the components with the length of the strips $f(x)$ & $g(x)$, we multiply them together to get $f(x)g(x)$. Instead of summing over the components, we do the analogous thing for functions; we integrate. So $f \cdot g = \int_a^b f(x) g(x)\,dx$.

In the same way that there exist orthogonal vectors, there are also orthogonal functions. With orthogonal vectors $\underline{u}_i \cdot \underline{u}_j = 0$ if $i \neq j$
$\neq 0$ if $i = j$

So with orthogonal functions $\int_a^b f(x) g(x)\,dx = 0$ if $f(x) \neq g(x)$
$\neq 0$ if $f(x) = g(x)$

We can make the functions orthonormal in the same way that we would for the vectors. With orthonormal vectors $\underline{u}_i \cdot \underline{u}_j = \delta_{ij} = 0$ if $i \neq j$
$= 1$ if $i = j$.

and we want $\int_a^b f(x) g(x)\,dx = \delta_{fg} = 0$ if $f \neq g$.
$= 1$ if $f = g$

So, if $\int_a^b f(x) f(x)\,dx = N$, then $\frac{1}{\sqrt{N}} f(x)$ is an orthonormal function because $\int_a^b \frac{f(x)}{\sqrt{N}} \frac{f(x)}{\sqrt{N}}\,dx = \frac{N}{N} = 1$. If we do this to each function in the set, then we will have $\int_a^b f(x) g(x)\,dx = \delta_{fg}$, so all the functions will be orthonormal.

Question 1

If $\underline{u}, \underline{v} \in IPS$ and $\underline{u}$ is perpendicular to $\underline{v}$ show that $\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$.

Solution

$$\|\underline{u}+\underline{v}\|^2 = (\underline{u}+\underline{v}) \cdot (\underline{u}+\underline{v}) = \underline{u} \cdot \underline{u} + \underline{v} \cdot \underline{v} + 2\underline{u} \cdot \underline{v} = \|\underline{u}\|^2 + \|\underline{v}\|^2 + 2\underline{u} \cdot \underline{v}$$

However, $\underline{u}$ and $\underline{v}$ are perpendicular, so $\underline{u} \cdot \underline{v} = 0$, which means that $\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$

Question 2

Using the normal dot product, for which values of $\underline{u}$ perpendicular to $\underline{v}$ a) $\underline{u} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} 1 \\ 7 \\ k \end{pmatrix}$ b) $\underline{u} = \begin{pmatrix} k \\ k \\ 1 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} k \\ 5 \\ 6 \end{pmatrix}$