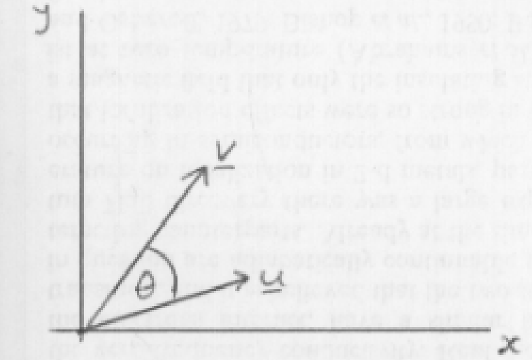
A vector space (or function space or matrix space) in which we have an inner product is called an inner product space (sometimes abbreviated to IPS).

The Cauchy-Schwartz Inequality

This is a mathematical idea that crops up frequently as a way to put a limit on a set of variables. It's often used to put an upper bound on errors when an approximation is made during a derivation.

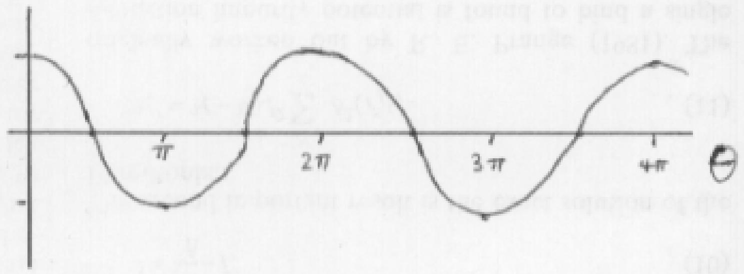
Imagine that we have two vectors in 2 dimensions.



Taking the dot product between them,

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \, \|\underline{v}\| \cos \theta$$

The function $\cos \theta$ looks like

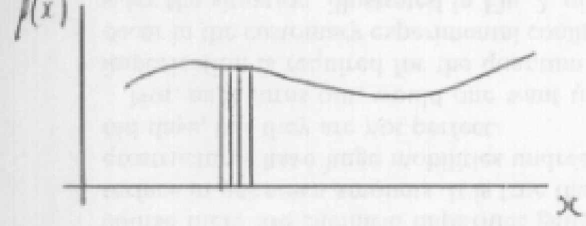


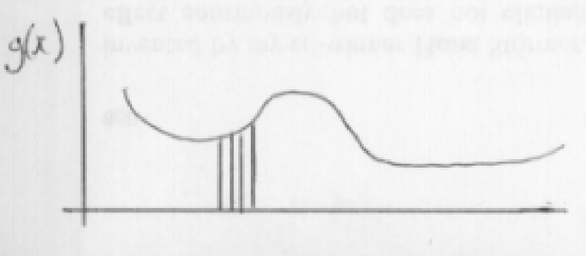
The highest point on the curve is 1, so $\cos \theta \leq 1$. Using this in the dot product, $\underline{u} \cdot \underline{v} \leq \|\underline{u}\| \, \|\underline{v}\|$. This is the Cauchy-Schwartz inequality.

The dot product for functions

We know what the dot product looks like for vectors. However, if we can apply the ideas from vector spaces to function spaces, it must follow that there is an equivalent to the dot product.

Suppose that we have two functions $f(x)$ and $g(x)$, analogous to the vectors $\underline{u}$ and $\underline{v}$. When we were calculating the area under curves, previously, we divided the area up into infinitely thin strips. The length of a strip located at x_0 is $f(x_0)$ in the upper diagram and $g(x_0)$ in the lower diagram.





$f(x)$ & $g(x)$ dont have components like $\underline{u}$ & $\underline{v}$. Instead, we use the length of the strips $f(x_0)$ and $g(x_0)$.

Now, let's look at the definition of the dot product.