B) The determinent of B is non-zero and the three columns are linearly independent, so the rank must be 3.

Coordinates and Bases

When we state the position of a point in 2 dimensions, we do so with a vector of the form $\begin{pmatrix} m \\ n \end{pmatrix}$. What this really means is $m\begin{pmatrix} 1 \\ 0 \end{pmatrix} + n\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, ie start at the origin and go m in the direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and then go n in the direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are an example of a set of basis vectors that span 2 dimensions. However, there is no reason why we can't use other sets of basis vectors, for example $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The question then occurs, "if a point is represented by $\begin{pmatrix} m \\ n \end{pmatrix}$ with the basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, how is it represented in the basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$?".

We can work it out by equating the two.

$$m\begin{pmatrix} 1 \\ 0 \end{pmatrix} + n\begin{pmatrix} 0 \\ 1 \end{pmatrix} = p\begin{pmatrix} 1 \\ 1 \end{pmatrix} + q\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Giving $m = p+q$, $n = p-q$. After some rearrangement, this gives $p = \frac{1}{2}(m+n)$ and $q = \frac{1}{2}(m-n)$ for the new coordinates p and q.

We can apply this idea to points in any number of dimensions.

Question 2

Find the coordinates of the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ relative to the bases $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$ of $\mathbb{R}^3$. Do the same for $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Solution

Let's use p, q, r as the coordinates of $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in the new basis.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = p\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + q\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + r\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

This gives 3 simultaneous equations.

① $0 = p + 2q + 3r$

② $0 = 2q + 3r$

③ $1 = 3r$

Rearranging ③ $r = \frac{1}{3}$. Substituting for r in ① and ② gives.

① $0 = p + 2q + 1$

② $0 = 2q + 1$