Tutorial 16

The rank of a matrix

If we have a $n \times n$ matrix and we think of its columns as vectors, each with n components, then the rank is defined as the number of linearly independent vectors.

For example,

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 10 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

The 3 columns of A are linearly independent, ie we can't construct one of the columns as a linear combination of the other two. Because we have 3 linearly independent columns, the rank is 3.

The columns of B are linearly dependent, because (third column) = (first column) + (second column), so the rank is not 3. However, the first and second columns are linearly independent of each other. So are the second and third columns and the first and third columns. Because we can find a set of 2 linearly independent columns, the rank is 2.

In C, all three columns are multiples of each other, so we can't even find a pair of linearly independent columns. The largest set of linearly independent columns that we can find contains only one column (it doesn't matter which one). Hence, the rank is <u>1</u>.

We have tackled this question by dividing the matrices into column vectors. However, we would have arrived at the same result if we had divided the matrices into rows and taken each row to be a vector.

Now, there is a more systematic way to find the rank, which uses the determinent. When we take the determinent of a matrix that contains linearly dependent column (or row) vectors, we get zero. When the columns (or rows) are independent, we get a non-zero answer. So we can calculate the rank by finding the largest 'sub-matrix' that has a non-zero determinent.

For example, suppose that we have the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 10 \end{pmatrix}$