For the denominator, we will workout the mass of the whole hemisphere, slice by slice.

Using cylindrical coordinates, the area of a slice located at z is $\int_0^{\sqrt{1-z^2}}\int_0^{2\pi} r\,d\theta\,dr = \int_0^{\sqrt{1-z^2}} [r\theta]_0^{2\pi}\,dr = \int_0^{\sqrt{1-z^2}} 2\pi r\,dr = [\pi r^2]_0^{\sqrt{1-z^2}}$

$= \pi(1-z^2)$.

The upper limit on the r integration, $\sqrt{1-z^2}$, can be obtained by rearranging the equation $x^2+y^2+z^2=1$ into the equation of a circle, $x^2+y^2=r^2$, where r is the radius. When we do this, we get that $x^2+y^2=1-z^2$, so the radius, $r=\sqrt{1-z^2}$.

The volume of each slice is $\pi(1-z^2)\,dz$. If ρ is the density, then the mass of the slice is $\rho\pi(1-z^2)\,dz$.

Summing over all the slices gives us the mass of the hemisphere

$$\int_0^1 \rho\pi(1-z^2)\,dz = \int_0^1 (\rho\pi - \rho\pi z^2)\,dz = \left[\rho\pi z - \tfrac{1}{3}\rho\pi z^3\right]_0^1 = \tfrac{2}{3}\pi\rho$$

This is the denominator in our expression for $\bar{z}$. To obtain the numerator, we multiply the mass of the slice by its z-coordinate and then sum over all the slices.

Multiplying by the z-coordinate gives $z\rho\pi(1-z^2)\,dz$.

Summing, $\int_0^1 z\rho\pi(1-z^2)\,dz = \int_0^1 (\pi\rho z - \rho\pi z^3)\,dz = \left[\tfrac{1}{2}\pi\rho z^2 - \tfrac{1}{4}\rho\pi z^4\right]_0^1$

$= \tfrac{1}{2}\pi\rho - \tfrac{1}{4}\pi\rho = \tfrac{1}{4}\pi\rho$.

$$\bar{z} = \frac{\tfrac{1}{4}\pi\rho}{\tfrac{2}{3}\pi\rho} = \frac{3}{8}$$

Hence the centre of gravity lies at $(0, 0, \tfrac{3}{8})$.